Econ 311:Statistical Methods and Interpretations

Chapter 9

Hypothesis Testing:
Single Population

Chapter Goals

After completing this chapter, you should be able to:

- Formulate null and alternative hypotheses for applications involving
  - a single population mean from a normal distribution
  - a single population proportion (large samples)
  - the variance of a normal distribution
- Formulate a decision rule for testing a hypothesis
- Know how to use the critical value and p-value approaches to test the null hypothesis (for both mean and proportion problems)
- Know what Type I and Type II errors are
- Assess the power of a test

What is a Hypothesis?

- A hypothesis is a claim (assumption) about a population parameter:

  - population mean

Example: The mean monthly cell phone bill of this city is μ = $42

  - population proportion

Example: The proportion of adults in this city with cell phones is $p = .68$

The Null Hypothesis, H_0

- States the assumption (numerical) to be tested

> Example: The average number of TV sets in U.S. Homes is equal to three ($H_0: \mu = 3$)

- Is always about a population parameter, not about a sample statistic

$$H_0: \mu = 3$$

$$H_0: \bar{X} = 3$$

The Null Hypothesis, H_0

(continued)

- Begin with the assumption that the null hypothesis is true
  - Similar to the notion of innocent until proven guilty

- Refers to the status quo
- Always contains "=" , "≤" or "≥" sign
- May or may not be rejected

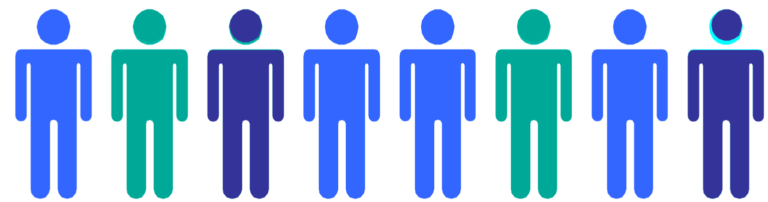
The Alternative Hypothesis, H_1

- Is the opposite of the null hypothesis
  - e.g., The average number of TV sets in U.S. homes is not equal to 3 (H_1: $\mu \neq 3$)
- Challenges the status quo
- Never contains the "=" , "≤" or "≥" sign
- May or may not be supported
- Is generally the hypothesis that the researcher is trying to support

Hypothesis Testing Process

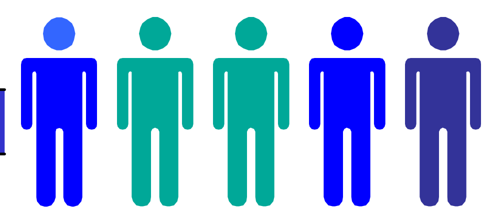
Claim: the population mean age is 50. (Null Hypothesis:

H_0: μ = 50)



Population

Now select a random sample



Sample

Is $\bar{X} = 20$ likely if μ = 50?

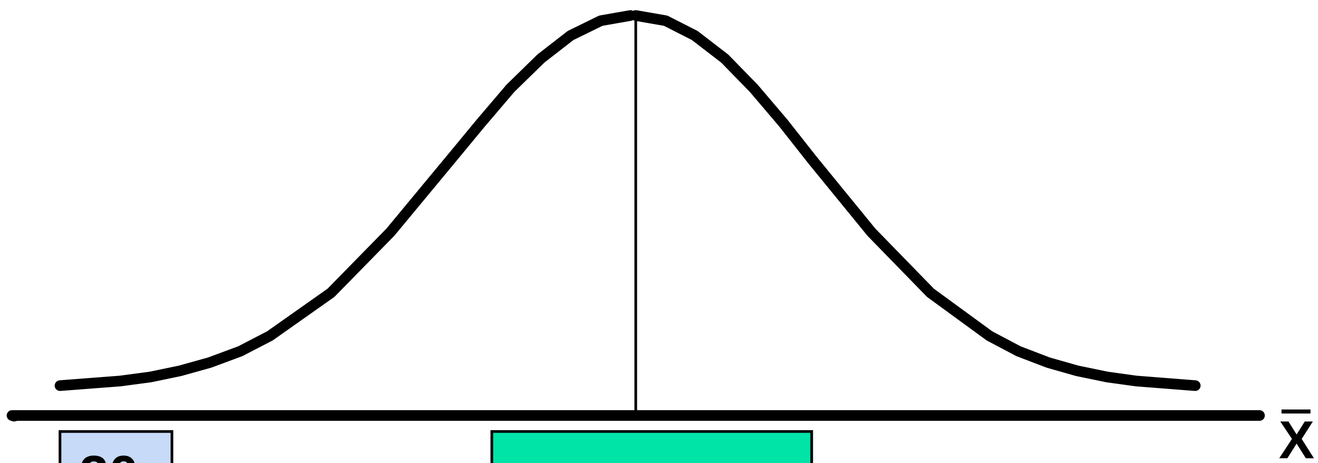
Suppose the sample mean age is 20: $\bar{X} = 20$

If not likely,

REJECT Null Hypothesis

Reason for Rejecting H_0

Sampling Distribution of $\bar{X}$



20

$\mu = 50$
If H_0 is true

If it is unlikely that we would get a sample mean of this value ...

... if in fact this were the population mean…

... then we reject the null hypothesis that $\mu = 50$.

Level of Significance, α

- **Defines the unlikely values of the sample statistic if the null hypothesis is true**
  - Defines rejection region of the sampling distribution
- Is designated by α , (level of significance)
  - Typical values are .01, .05, or .10
- Is selected by the researcher at the beginning
- Provides the critical value(s) of the test

Level of Significance and the Rejection Region

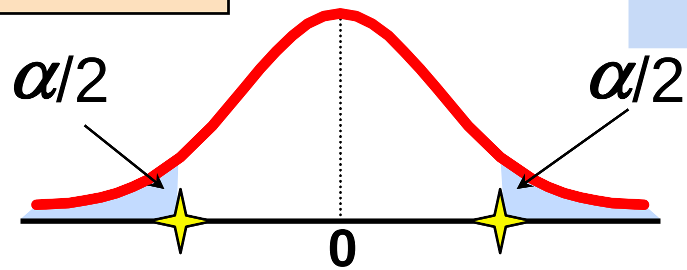
Level of significance = α

Represents critical value

Rejection region is shaded

H_0: $\mu = 3$

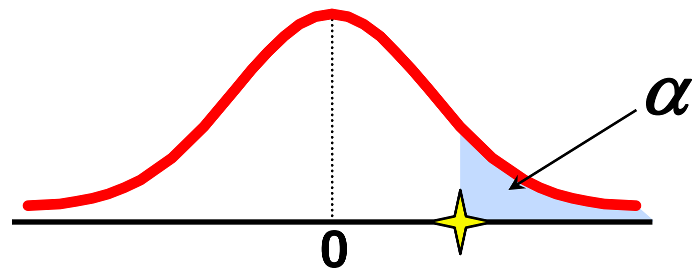
H_1: $\mu \neq 3$



Two-tail test

H_0: $\mu \leq 3$

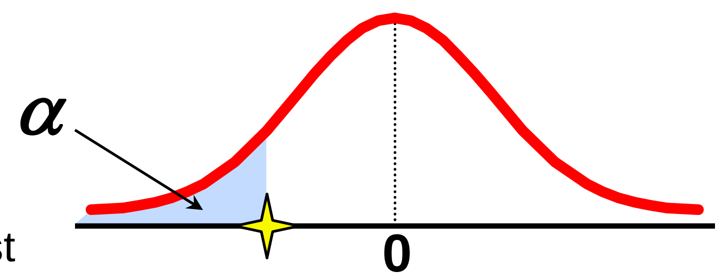
H_1: $\mu > 3$



Upper-tail test

H_0: $\mu \geq 3$

H_1: $\mu < 3$



Lower-tail test

Errors in Making Decisions

- **Type I Error**
  - Reject a true null hypothesis
  - Considered a serious type of error

The probability of Type I Error is α

- Called level of significance of the test
- Set by researcher in advance

Errors in Making Decisions

(continued)

- **Type II Error**
  - Fail to reject a false null hypothesis

The probability of Type II Error is β

Outcomes and Probabilities

Possible Hypothesis Test Outcomes

	Actual Situation	
Decision	H_0 True	H_0 False
Do Not Reject **H_0**	**No Error** **(1 - α)**	**Type II Error** **(β)**
Reject **H_0**	**Type I Error** **(α)**	**No Error** **(1 - β)**

Key:
Outcome
(Probability)

Type I & II Error Relationship

- Type I and Type II errors can not happen at the same time
  - Type I error can only occur if H_0 is true
  - Type II error can only occur if H_0 is false

> If Type I error probability (α) ↑, then
>
> Type II error probability (β) ↓

Factors Affecting Type II Error

- All else equal,
  - β ↑ when the difference between hypothesized parameter and its true value ↓
  - β ↑ when α ↓
  - β ↑ when σ ↑
  - β ↑ when *n* ↓

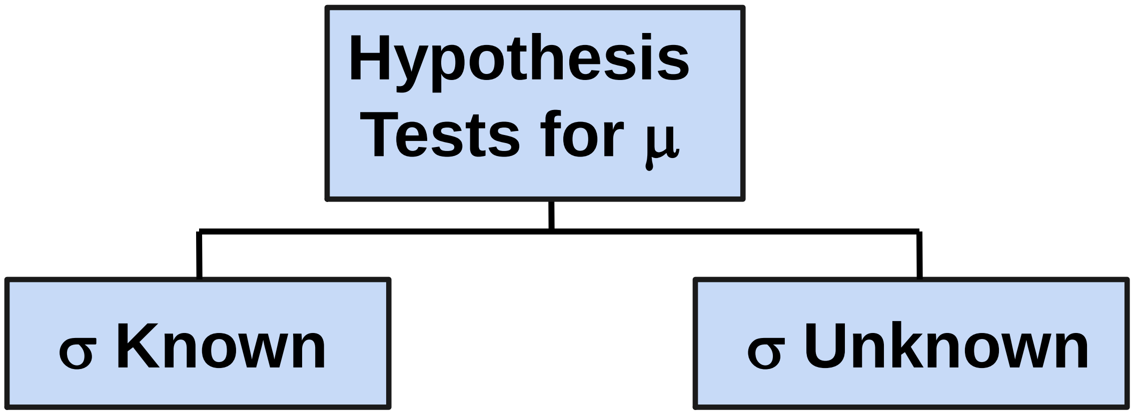
Power of the Test

- The power of a test is the probability of rejecting a null hypothesis that is false

- i.e., Power = P(Reject H_0 | H_1 is true)

  - Power of the test increases as the sample size increases

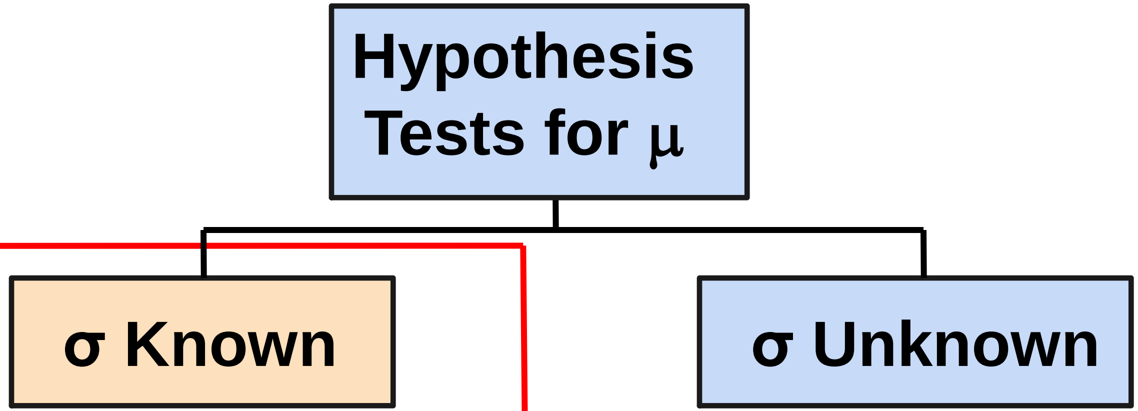
Hypothesis Tests for the Mean



Test of Hypothesis for the Mean (σ Known)

9.2

- Convert sample result ($\bar{x}$) to a z value



Consider the test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0$$

(Assume the population is normal)

The decision rule is:

$$\text{Reject } H_0 \text{ if } z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} > z_\alpha$$

Decision Rule

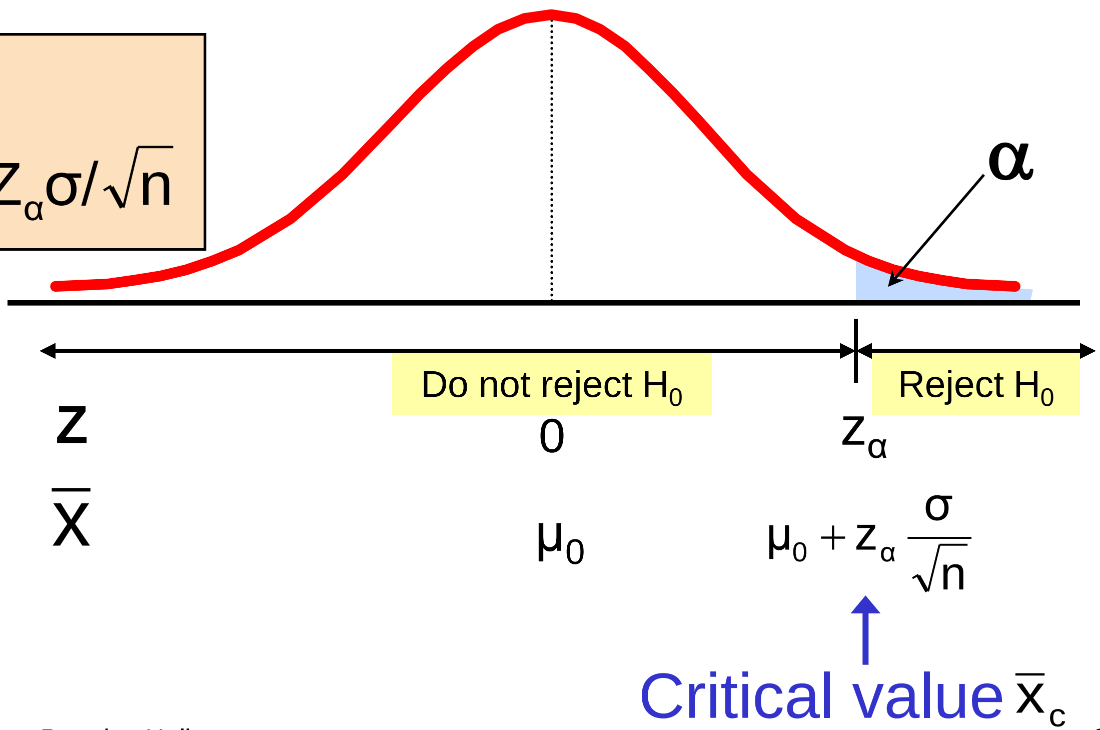
Reject H_0 if $z = \dfrac{\bar{x} - \mu_0}{\dfrac{\sigma}{\sqrt{n}}} > z_\alpha$

H_0: $\mu = \mu_0$

H_1: $\mu > \mu_0$

Alternate rule:

Reject H_0 if $\bar{x} > \mu_0 + Z_\alpha \sigma/\sqrt{n}$



p-Value Approach to Testing

- p-value: Probability of obtaining a test statistic more extreme (≤ or ≥) than the observed sample value given H_0 is true
  - Also called observed level of significance
  - Smallest value of α for which H_0 can be rejected

p-Value Approach to Testing

(continued)

- Convert sample result (e.g., $\bar{x}$) to test statistic (e.g., z statistic)
- Obtain the p-value
  - For an upper tail test:

$$\text{p-value} = P\left(z > \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}\text{, given that } H_0 \text{ is true}\right)$$

$$= P\left(z > \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \mid \mu = \mu_0\right)$$

- Decision rule: compare the p-value to α
  - If p-value $< \alpha$, reject H_0
  - If p-value $\geq \alpha$, do not reject H_0

Example: Upper-Tail Z Test for Mean (σ Known)

A phone industry manager thinks that customer monthly cell phone bill have increased, and now average over \$52 per month. The company wishes to test this claim. (Assume $\sigma = 10$ is known)

Form hypothesis test:

H_0: $\mu \leq 52$ the average is not over \$52 per month

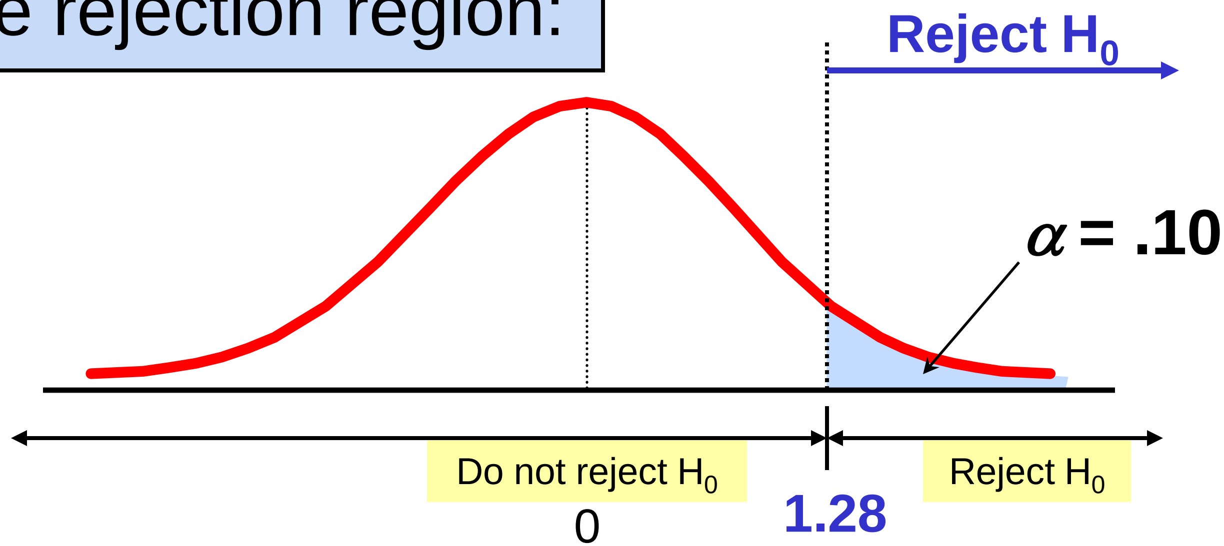
H_1: $\mu > 52$ the average is greater than \$52 per month (i.e., sufficient evidence exists to support the manager's claim)

Example: Find Rejection Region

(continued)

- Suppose that α = .10 is chosen for this test

Find the rejection region:



Reject H_0 if $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} > 1.28$

Example: Sample Results

(continued)

Obtain sample and compute the test statistic

Suppose a sample is taken with the following results: $n = 64$, $\bar{x} = 53.1$ ($\sigma = 10$ was assumed known)

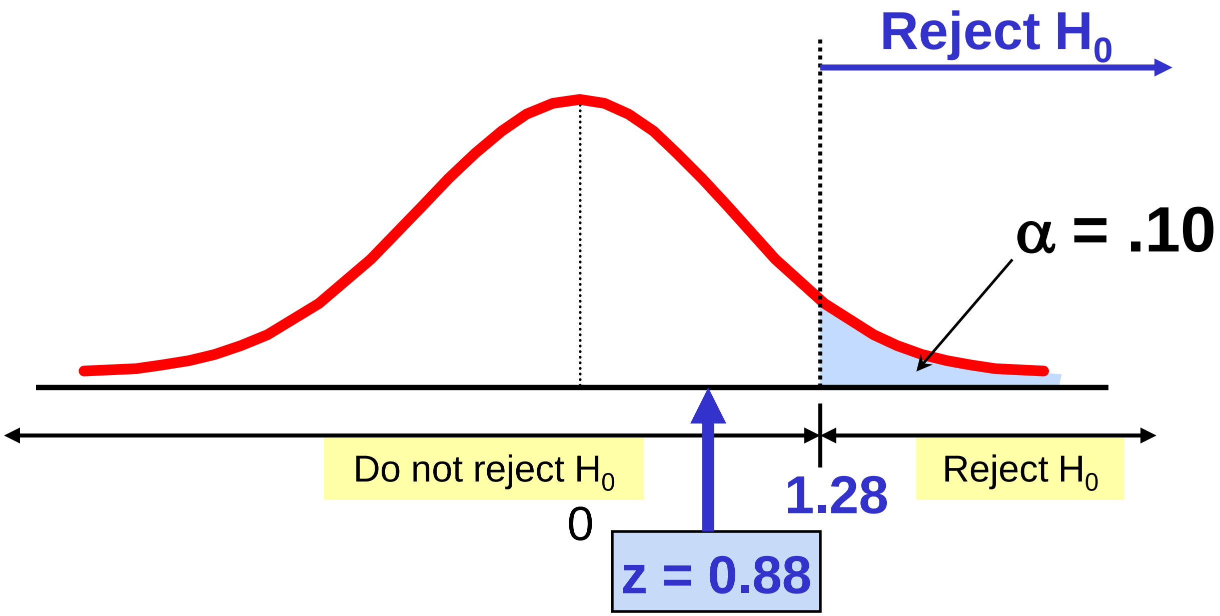
- Using the sample results,

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{53.1 - 52}{\frac{10}{\sqrt{64}}} = 0.88$$

Example: Decision

(continued)

Reach a decision and interpret the result:



Do not reject H_0 since z = 0.88 < 1.28

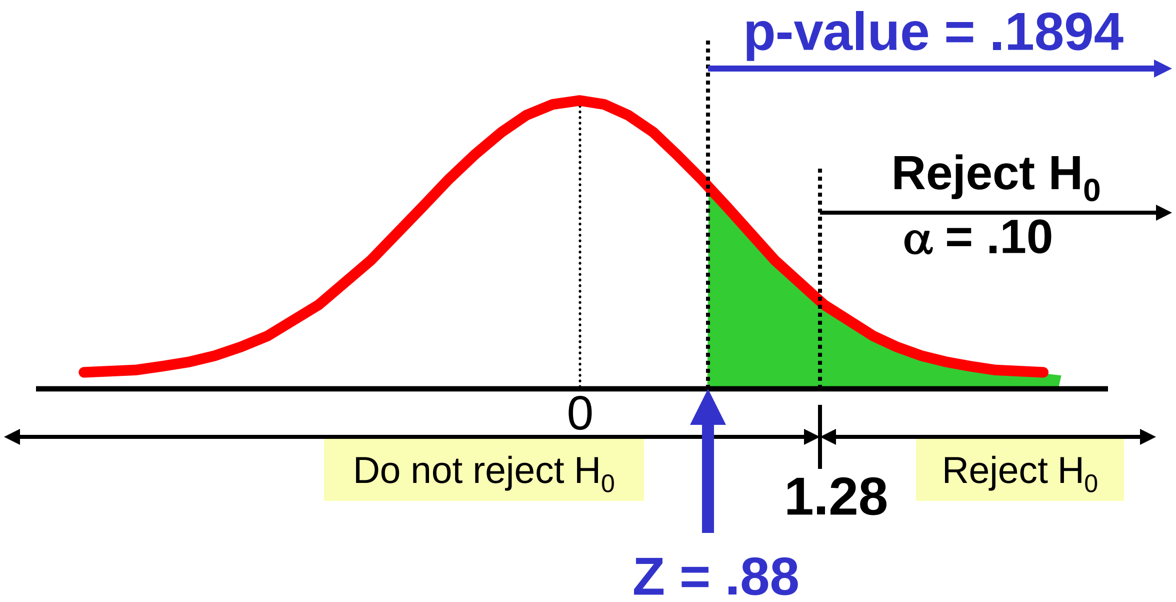
i.e.: there is not sufficient evidence that the mean bill is over $52

Example: p-Value Solution

(continued)

Calculate the p-value and compare to α

(assuming that μ = 52.0)



$$P(\bar{x} \geq 53.1 \mid \mu = 52.0)$$

$$= P\left(z \geq \frac{53.1 - 52.0}{10/\sqrt{64}} \right)$$

$$= P(z \geq 0.88) = 1 - .8106$$

$$= .1894$$

Do not reject H_0 since p-value = .1894 > α = .10

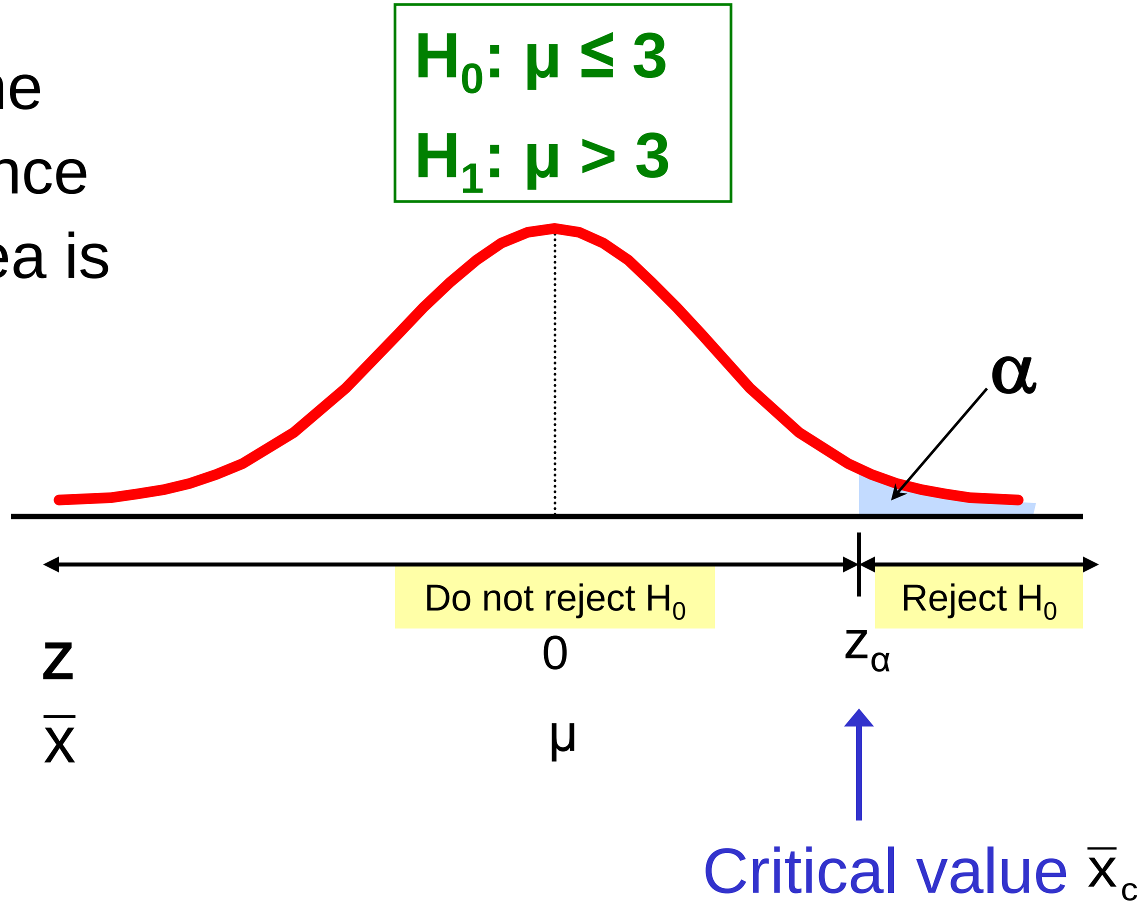
One-Tail Tests

- In many cases, the alternative hypothesis focuses on one particular direction

H_0: $\mu \le 3$ H_1: $\mu > 3$	This is an upper-tail test since the alternative hypothesis is focused on the upper tail above the mean of 3
H_0: $\mu \ge 3$ H_1: $\mu < 3$	This is a lower-tail test since the alternative hypothesis is focused on the lower tail below the mean of 3

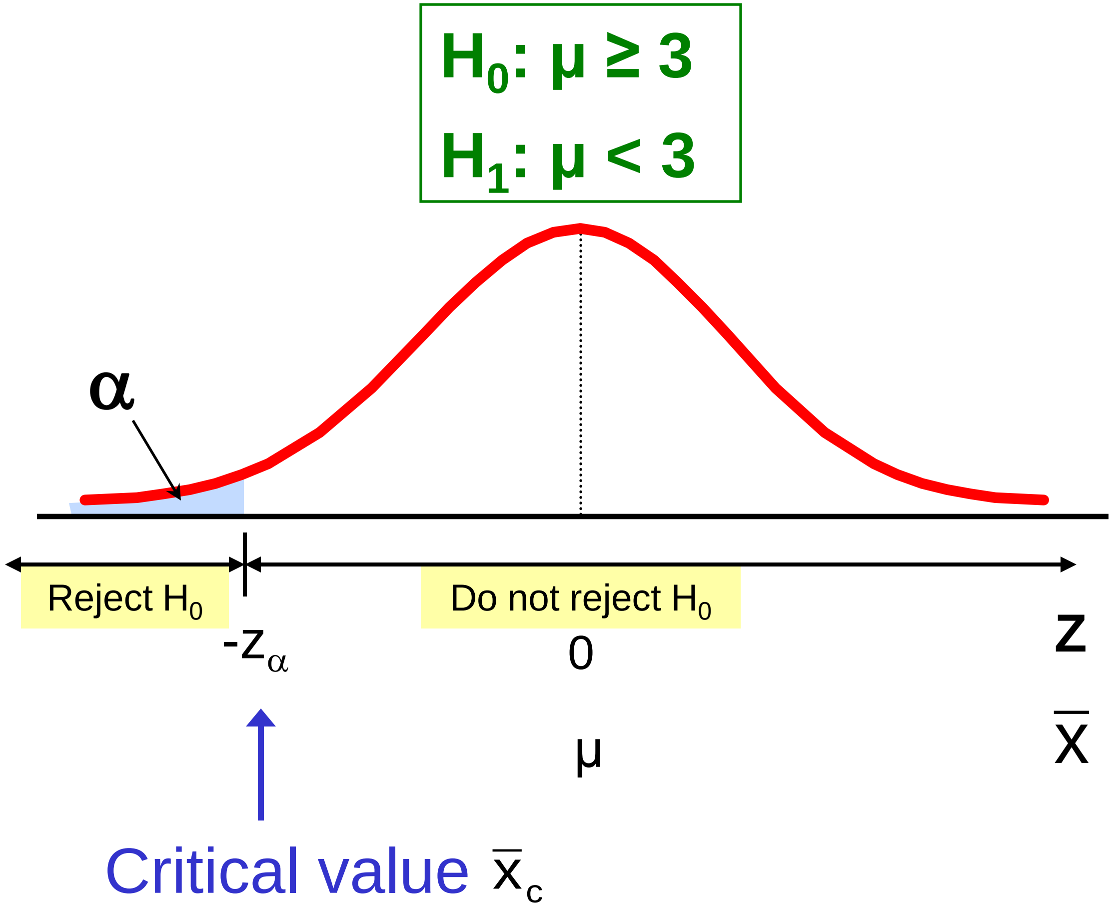
Upper-Tail Tests

- There is only one critical value, since the rejection area is in only one tail



Lower-Tail Tests

- There is only one critical value, since the rejection area is in only one tail

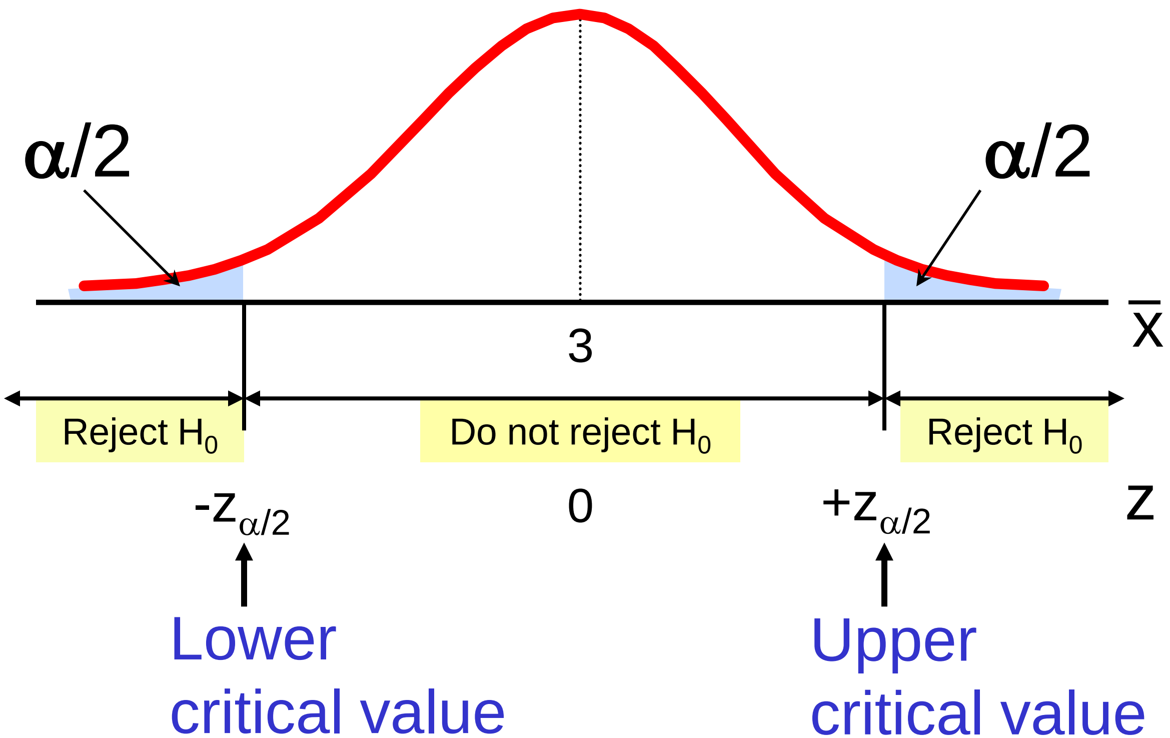


Two-Tail Tests

- In some settings, the alternative hypothesis does not specify a unique direction

$H_0: \mu = 3$
$H_1: \mu \neq 3$

- There are two critical values, defining the two regions of rejection



Hypothesis Testing Example

Test the claim that the true mean # of TV sets in US homes is equal to 3.
(Assume σ = 0.8)

- State the appropriate null and alternative hypotheses
  - H_0: μ = 3 , H_1: μ ≠ 3 (This is a two tailed test)
- Specify the desired level of significance
  - Suppose that α = .05 is chosen for this test
- Choose a sample size
  - Suppose a sample of size n = 100 is selecte

Hypothesis Testing Example

(continued)

- Determine the appropriate technique
  - σ is known so this is a z test
- Set up the critical values
  - For $\alpha = .05$ the critical z values are ±1.96
- Collect the data and compute the test statistic
  - Suppose the sample results are

    $n = 100$, $\bar{x} = 2.84$ ($\sigma = 0.8$ is assumed known)

So the test statistic is:

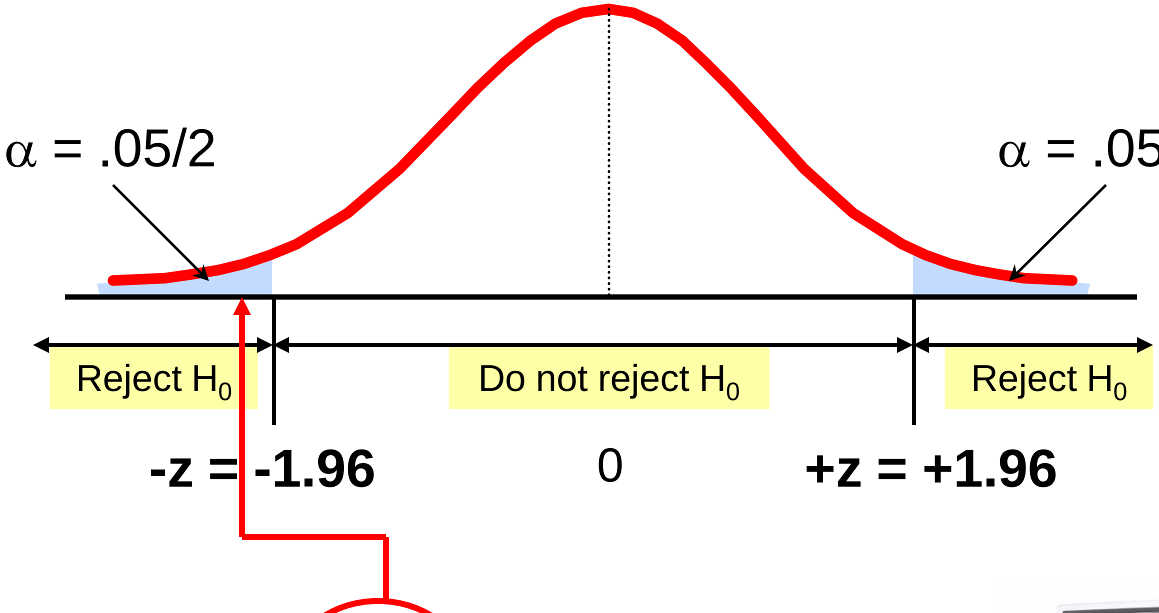
$$z = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{2.84 - 3}{\frac{0.8}{\sqrt{100}}} = \frac{-.16}{.08} = -2.0$$

Hypothesis Testing Example

(continued)

- Is the test statistic in the rejection region?

Reject H_0 if $z < -1.96$ or $z > 1.96$; otherwise do not reject H_0

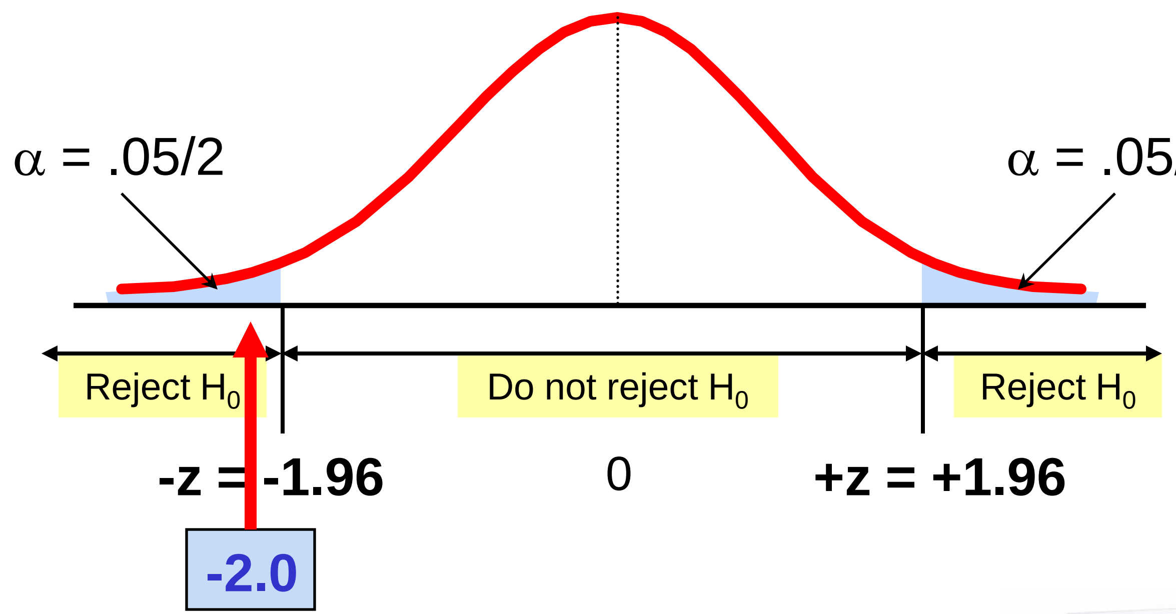


Here, $z = -2.0 < -1.96$, so the test statistic is in the rejection region

Hypothesis Testing Example

(continued)

- Reach a decision and interpret the result



Since $z = -2.0 < -1.96$, we reject the null hypothesis and conclude that there is sufficient evidence that the mean number of TVs in US homes is not equal to 3

Example: p-Value

- Example: How likely is it to see a sample mean of 2.84 (or something further from the mean, in either direction) if the true mean is μ = 3.0?

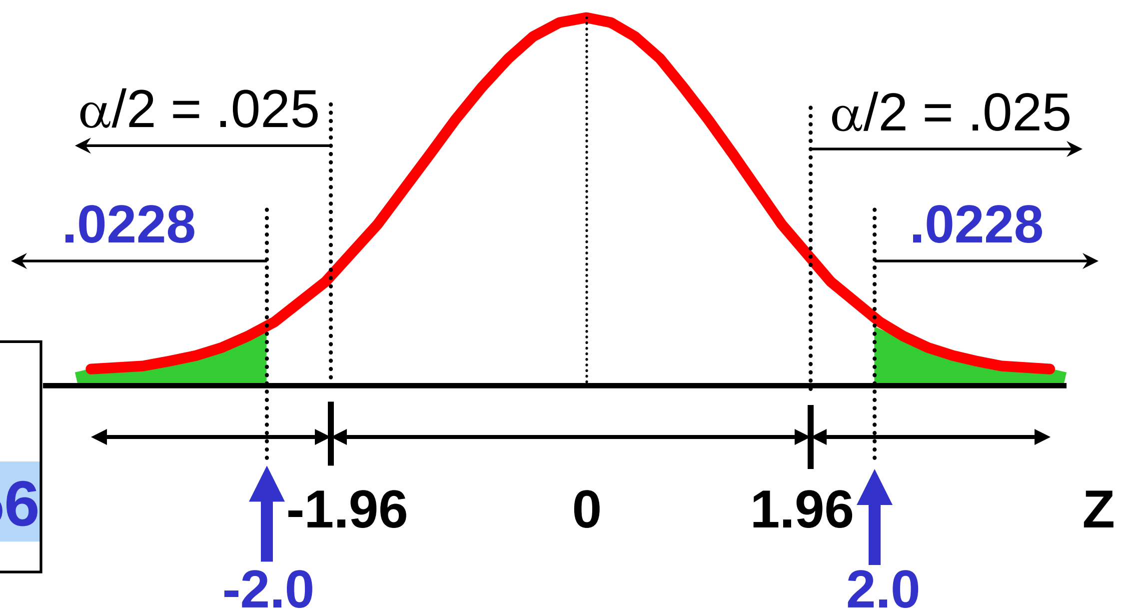
$\bar{x}$ = 2.84 is translated to a z score of z = -2.0

$$P(z < -2.0) = .0228$$

$$P(z > 2.0) = .0228$$

p-value

= .0228 + .0228 = .0456



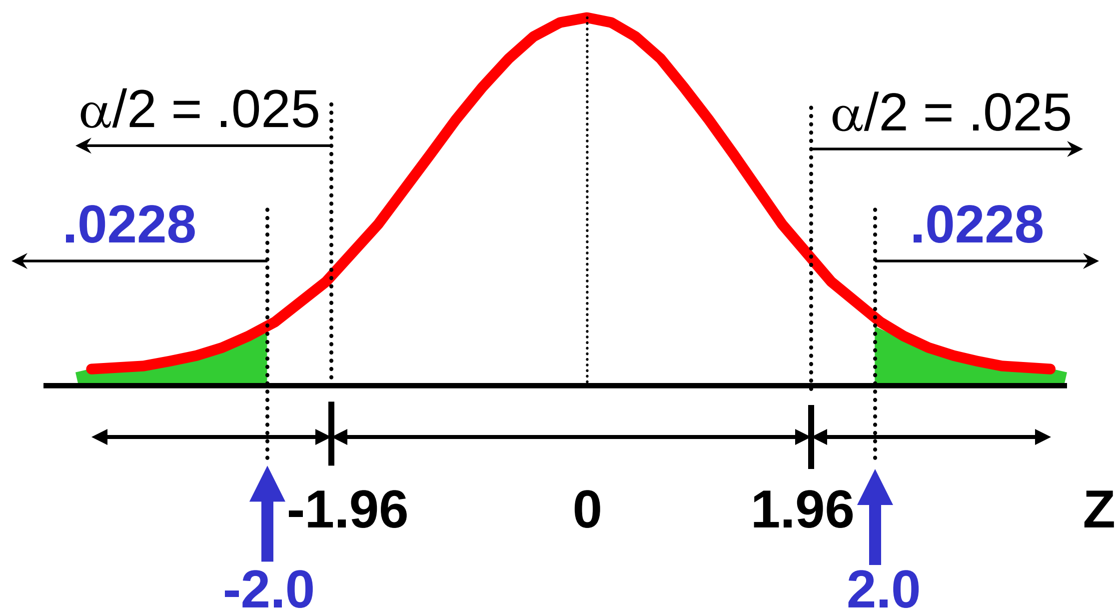
Example: p-Value

(continued)

- Compare the p-value with α
  - If p-value < α , reject H_0
  - If p-value ≥ α , do not reject H_0

Here: p-value = .0456
α = .05

Since .0456 < .05, we reject the null hypothesis



9.3

t Test of Hypothesis for the Mean (σ Unknown)

- Convert sample result ($\bar{X}$) to a t test statistic

Hypothesis Tests for μ

σ Known | **σ Unknown**

Consider the test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0$$

(Assume the population is normal)

The decision rule is:

$$\text{Reject } H_0 \text{ if } t = \frac{\bar{X} - \mu_0}{\frac{S}{\sqrt{n}}} > t_{n-1,\, \alpha}$$

t Test of Hypothesis for the Mean (σ Unknown)

(continued)

- For a two-tailed test:

  Consider the test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0$$

(Assume the population is normal, and the population variance is unknown)

The decision rule is:

Reject H_0 if $t = \dfrac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} < -t_{n-1, \alpha/2}$ or if $t = \dfrac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} > t_{n-1, \alpha/2}$

Example: Two-Tail Test (σ Unknown)

The average cost of a hotel room in Chicago is said to be $168 per night. A random sample of 25 hotels resulted in $\bar{x}$ = $172.50 and s = $15.40. Test at the α = 0.05 level.

(Assume the population distribution is normal)

H_0: $\mu = 168$

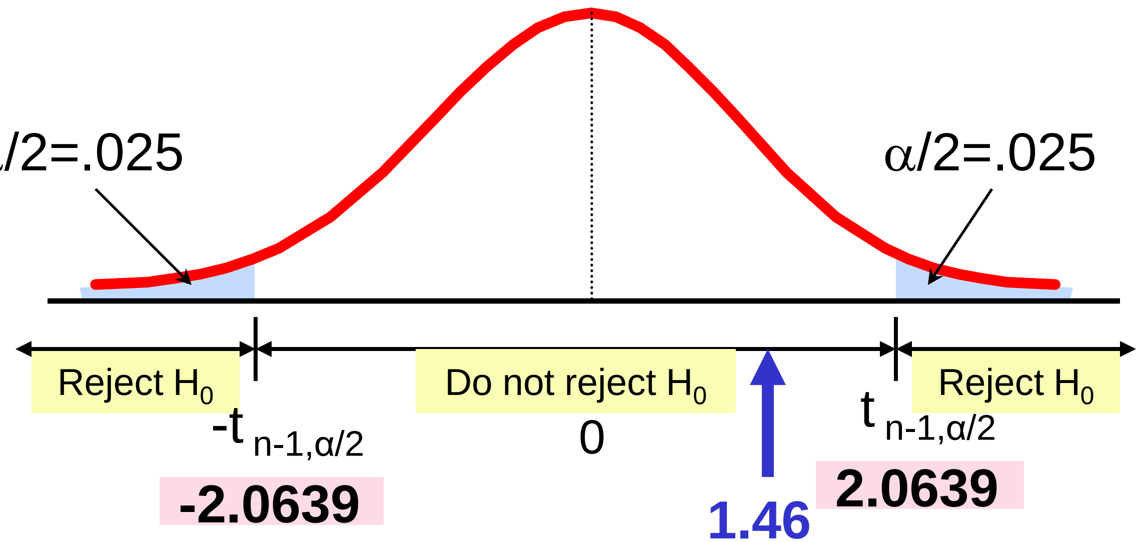
H_1: $\mu \neq 168$

Example Solution: Two-Tail Test

H_0: μ = 168
H_1: μ ≠ 168

- α = 0.05
- n = 25
- σ is unknown, so use a **t statistic**
- Critical Value:

$t_{24\,,\,.025} = \pm\ 2.0639$



$$t_{n-1} = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{172.50 - 168}{\frac{15.40}{\sqrt{25}}} = 1.46$$

Do not reject H_0: not sufficient evidence that true mean cost is different than $168

9.4

Tests of the Population Proportion

- Involves categorical variables
- Two possible outcomes
  - "Success" (a certain characteristic is present)
  - "Failure" (the characteristic is not present)
- Fraction or proportion of the population in the "success" category is denoted by P
- Assume sample size is large

Proportions

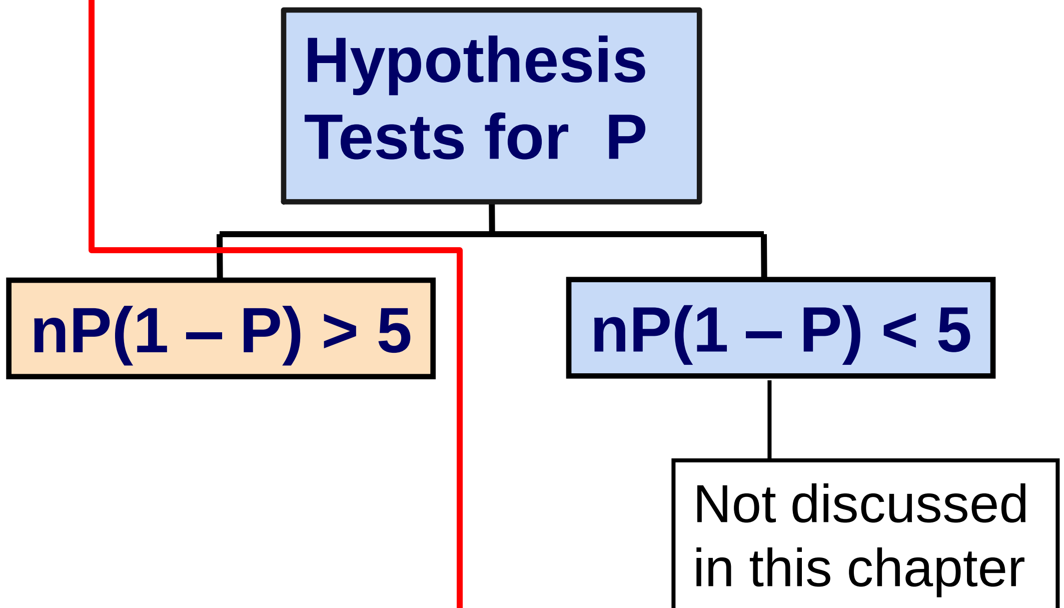
(continued)

- Sample proportion in the success category is denoted by $\hat{p}$
  - $$\hat{p} = \frac{x}{n} = \frac{\text{number of successes in sample}}{\text{sample size}}$$
- When $nP(1 - P) > 5$, $\hat{p}$ can be approximated by a normal distribution with mean and standard deviation
  - $$\mu_{\hat{p}} = P \qquad \sigma_{\hat{p}} = \sqrt{\frac{P(1-P)}{n}}$$

Hypothesis Tests for Proportions

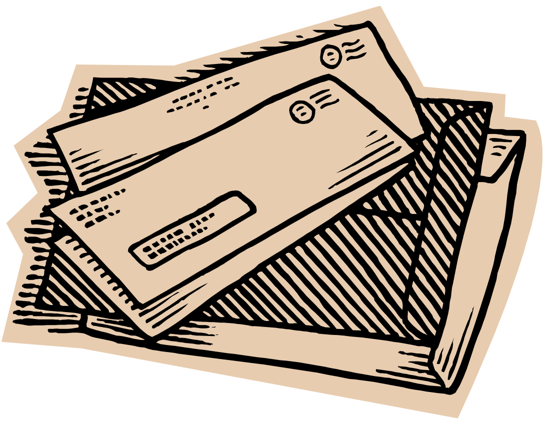
- The sampling distribution of $\hat{p}$ is approximately normal, so the test statistic is a z value:

$$z = \frac{\hat{p} - P_0}{\sqrt{\frac{P_0(1 - P_0)}{n}}}$$



Example: Z Test for Proportion

A marketing company claims that it receives 8% responses from its mailing. To test this claim, a random sample of 500 were surveyed with 25 responses. Test at the $\alpha = .05$ significance level.



Check:

Our approximation for P is

$\hat{p} = 25/500 = .05$

$nP(1 - P) = (500)(.05)(.95)$
$= 23.75 > 5$ ✓

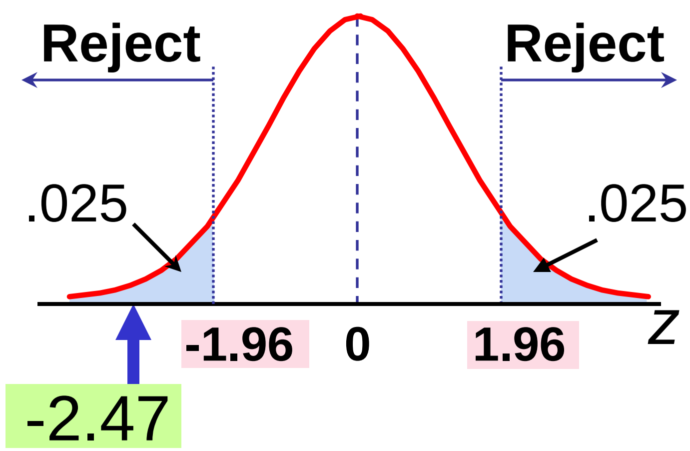
Z Test for Proportion: Solution

H_0: P = .08
H_1: P ≠ .08

$\alpha = .05$

$n = 500, \quad \hat{p} = .05$

Critical Values: ± 1.96



Test Statistic:

$$z = \frac{\hat{p} - P_0}{\sqrt{\frac{P_0(1-P_0)}{n}}} = \frac{.05 - .08}{\sqrt{\frac{.08(1-.08)}{500}}} = -2.47$$

Decision:

Reject H_0 at $\alpha = .05$

Conclusion:

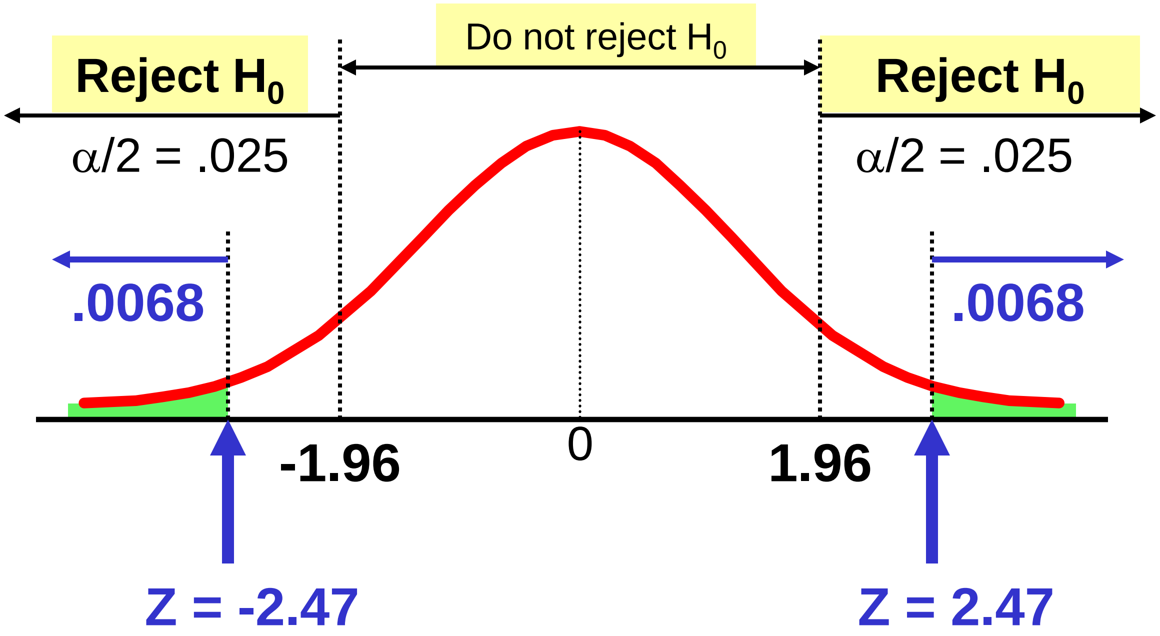
There is sufficient evidence to reject the company's claim of 8% response rate.

p-Value Solution

(continued)

Calculate the p-value and compare to α

(For a two sided test the p-value is always two sided)



p-value = .0136:

$$P(Z \le -2.47) + P(Z \ge 2.47)$$

$$= 2(.0068) = 0.0136$$

Reject H_0 since p-value = .0136 < α = .05

Power of the Test

- Recall the possible hypothesis test outcomes:

	Actual Situation	
Decision	H_0 True	H_0 False
Do Not Reject H_0	**No error** **(1 - α)**	**Type II Error** **(β)**
Reject H_0	**Type I Error** **(α)**	**No Error** **(1 - β)**

Key:
Outcome
(Probability)

- β denotes the probability of Type II Error
- 1 – β is defined as the power of the test

Power = 1 – β = the probability that a false null hypothesis is rejected

Type II Error

Assume the population is normal and the population variance is known. Consider the test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0$$

The decision rule is:

$$\text{Reject } H_0 \text{ if } \; z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} > z_\alpha$$

or

$$\text{Reject } H_0 \text{ if } \; \bar{x} = \bar{x}_c > \mu_0 + Z_\alpha \sigma / \sqrt{n}$$

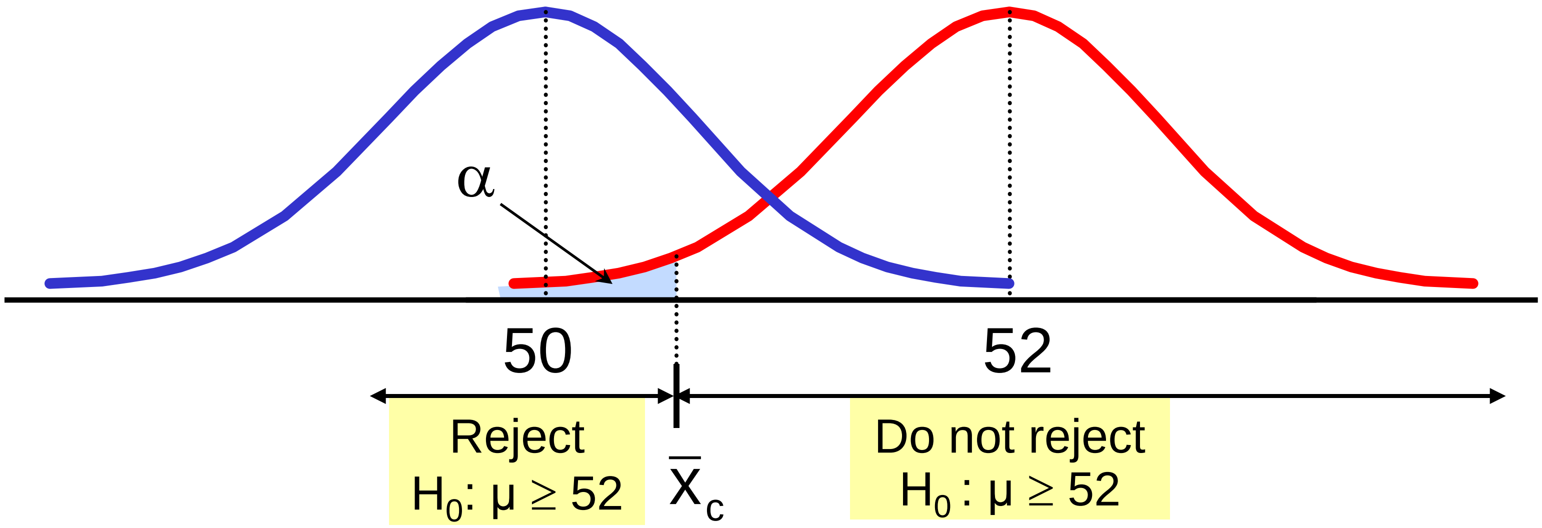
If the null hypothesis is false and the true mean is μ*, then the probability of type II error is

$$\beta = P(\bar{x} < \bar{x}_c \mid \mu = \mu^*) = P\left(z < \frac{\bar{x}_c - \mu^*}{\sigma / \sqrt{n}} \right)$$

Type II Error Example

- Type II error is the probability of failing to reject a false H_0

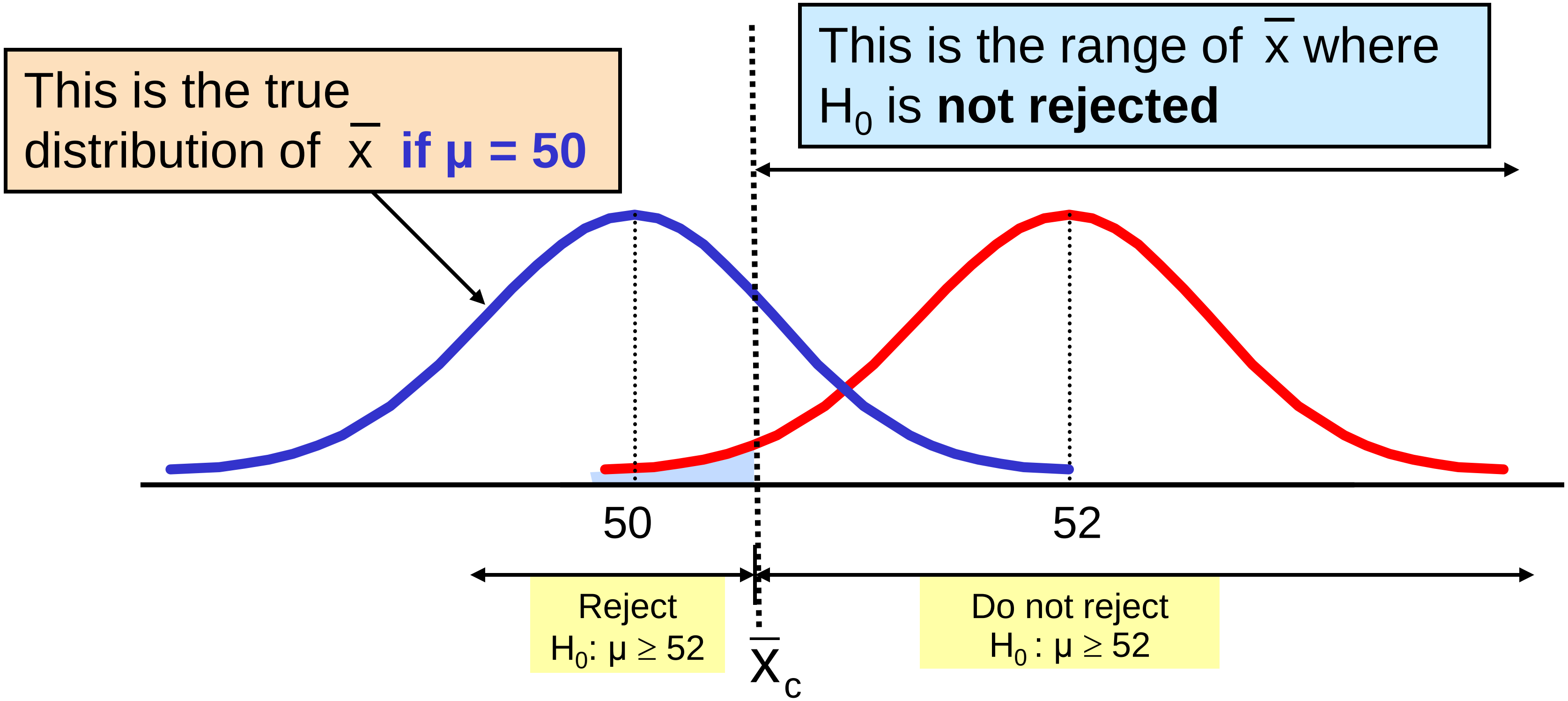
Suppose we fail to reject **H_0: $\mu \geq 52$**
when in fact the true mean is **$\mu^* = 50$**



Type II Error Example

(continued)

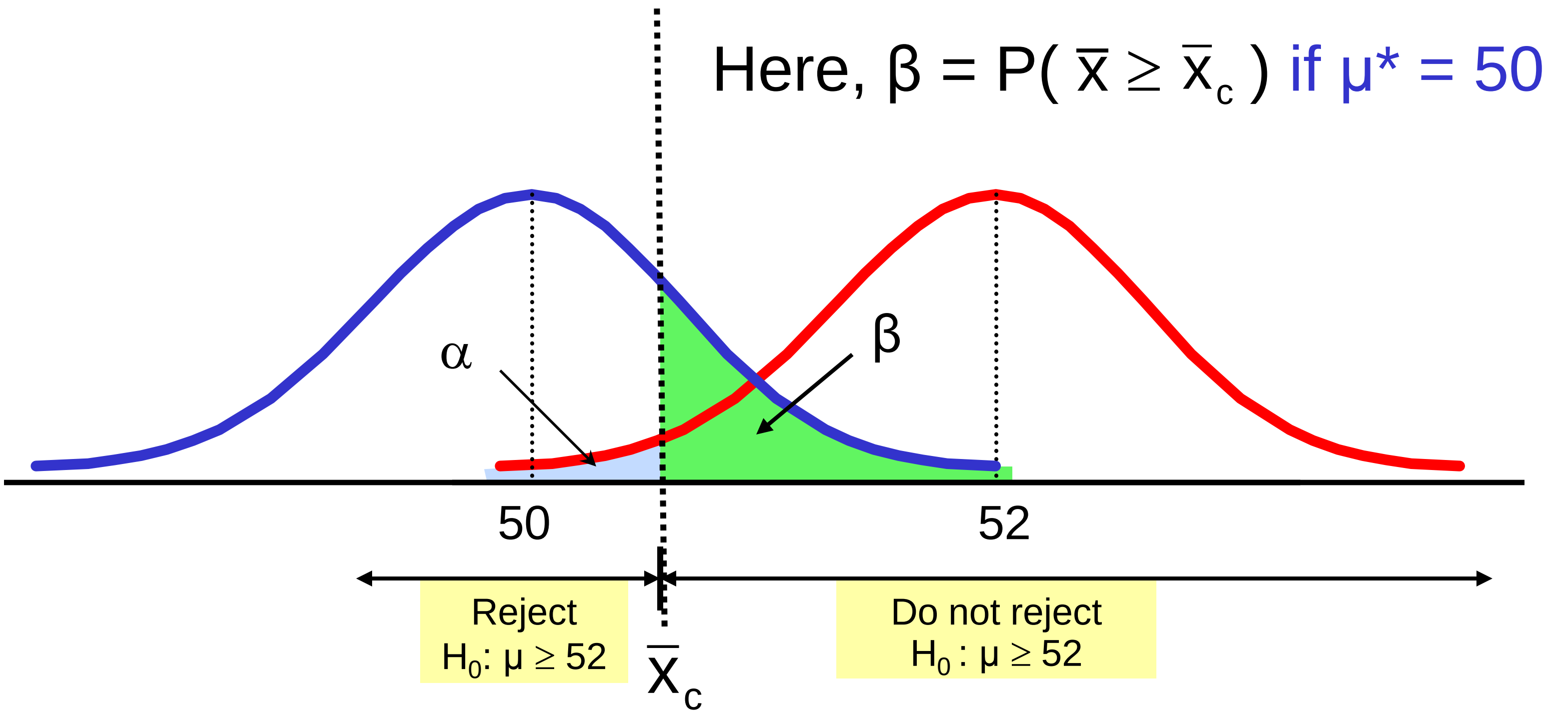
- Suppose we do not reject H_0: $\mu \geq 52$ when in fact the true mean is $\mu^* = 50$



Type II Error Example

(continued)

- Suppose we do not reject $H_0: \mu \geq 52$ when in fact the true mean is $\mu^* = 50$



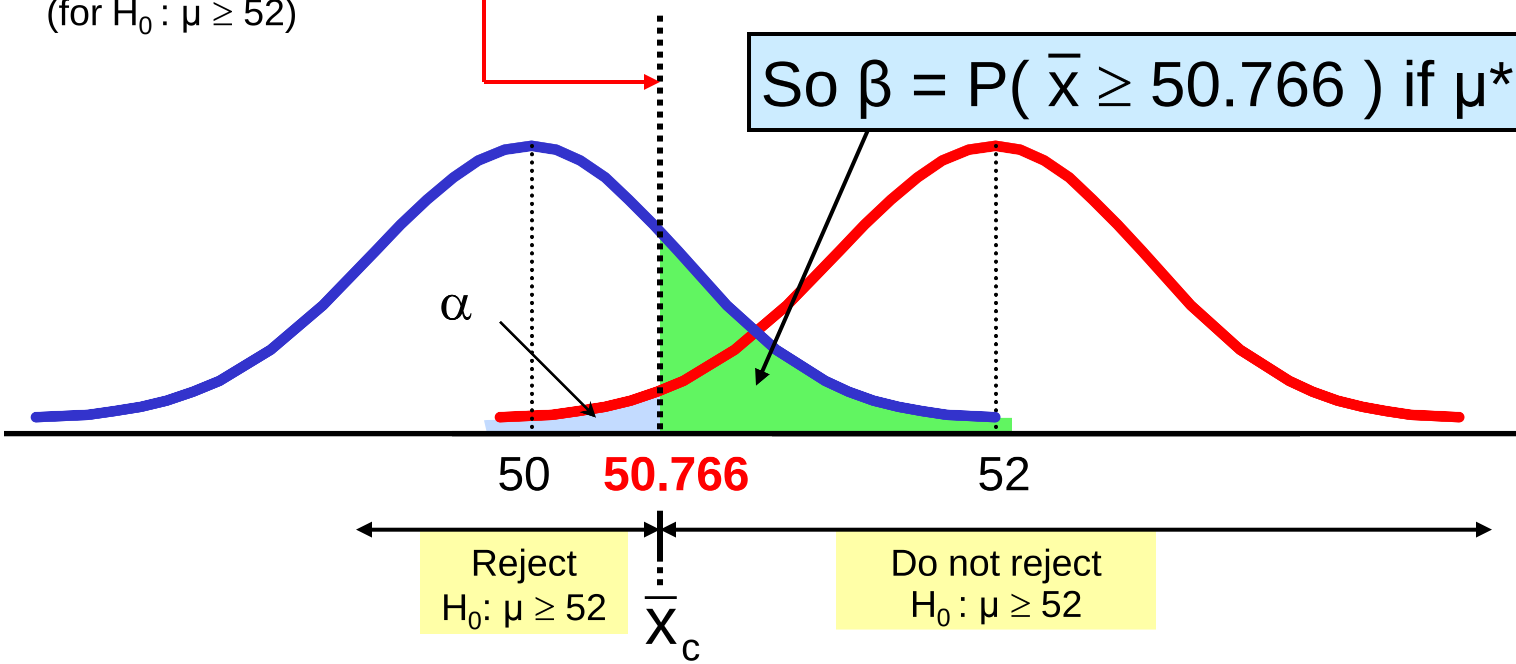
Calculating β

- Suppose n = 64 , σ = 6 , and α = .05

$$\bar{x}_c = \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} = 52 - 1.645 \frac{6}{\sqrt{64}} = 50.766$$

(for $H_0 : \mu \ge 52$)

So $\beta = P(\bar{x} \ge 50.766)$ if $\mu^* = 50$

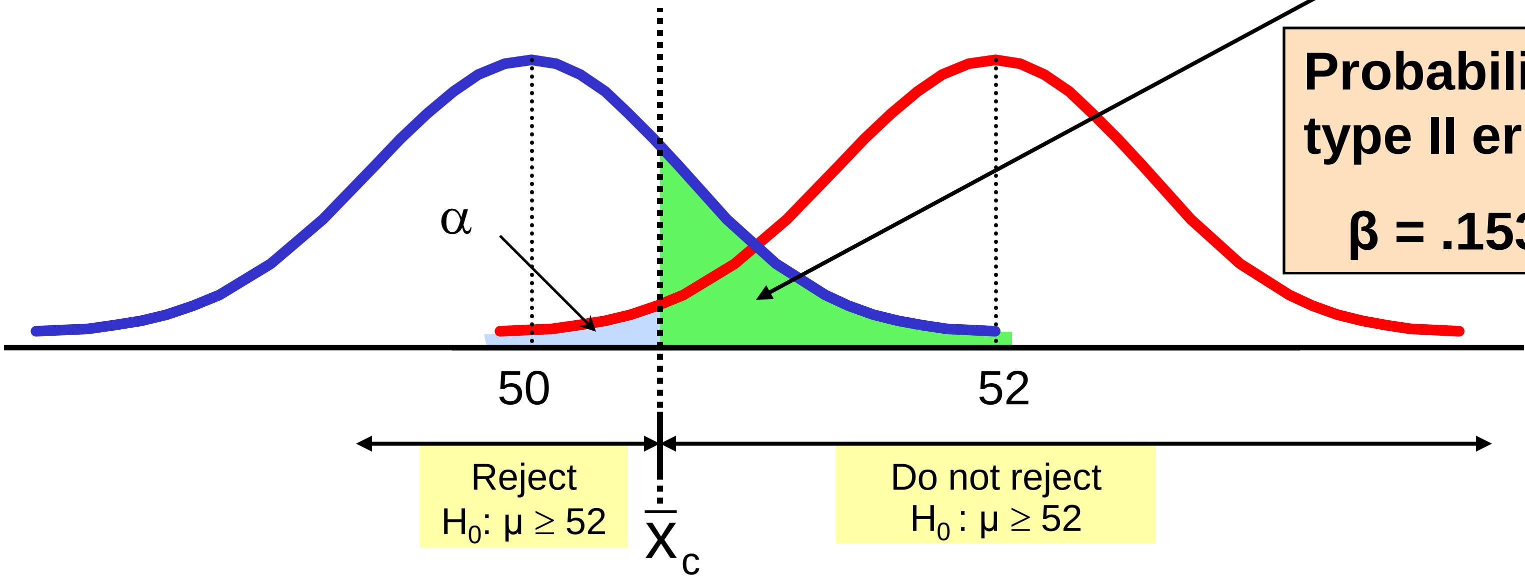


Calculating β

(continued)

- Suppose n = 64 , σ = 6 , and α = .05

$$P(\bar{x} \geq 50.766 \mid \mu^* = 50) = P\left(z \geq \frac{50.766 - 50}{6/\sqrt{64}} \right) = P(z \geq 1.02) = .5 - .3461 = .1539$$



Probability of type II error:

β = .1539

Power of the Test Example

If the true mean is $\mu^* = 50$,

- The probability of Type II Error = $\beta = 0.1539$
- The power of the test = $1 - \beta = 1 - 0.1539 = 0.8461$

Key:
Outcome
(Probability)

	Actual Situation	
Decision	H_0 True	H_0 False
Do Not Reject H_0	**No error** **$1 - \alpha = 0.95$**	**Type II Error** **$\beta = 0.1539$**
Reject H_0	**Type I Error** **$\alpha = 0.05$**	**No Error** **$1 - \beta = 0.8461$**

(The value of β and the power will be different for each μ^*)

Hypothesis Tests of one Population Variance

9.6

- Goal: Test hypotheses about the population variance, σ^2

  - If the population is normally distributed,

$$\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma^2}$$

has a chi-square distribution with (n – 1) degrees of freedom

Hypothesis Tests of one Population Variance

(continued)

The test statistic for hypothesis tests about one population variance is

$$\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma_0^2}$$

Decision Rules: Variance

Population variance

Lower-tail test:

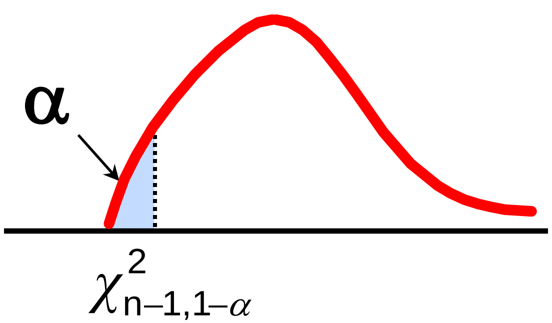
$H_0: \sigma^2 \geq \sigma_0^2$
$H_1: \sigma^2 < \sigma_0^2$

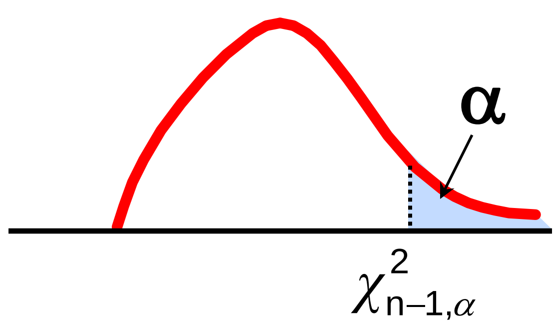
Upper-tail test:

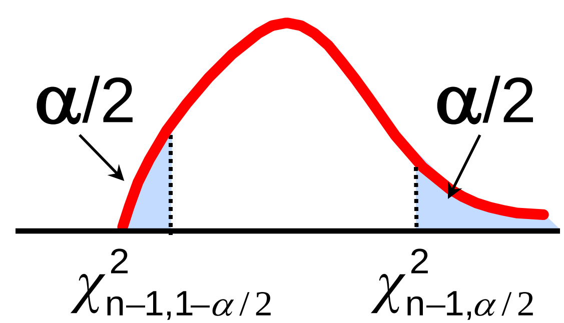
$H_0: \sigma^2 \leq \sigma_0^2$
$H_1: \sigma^2 > \sigma_0^2$

Two-tail test:

$H_0: \sigma^2 = \sigma_0^2$
$H_1: \sigma^2 \neq \sigma_0^2$







Reject H_0 if
$\chi^2_{n-1} < \chi^2_{n-1,1-\alpha}$

Reject H_0 if
$\chi^2_{n-1} > \chi^2_{n-1,\alpha}$

Reject H_0 if
$\chi^2_{n-1} > \chi^2_{n-1,\alpha/2}$
or
$\chi^2_{n-1} < \chi^2_{n-1,1-\alpha/2}$

Chapter Summary

- Addressed hypothesis testing methodology
- Performed Z Test for the mean (σ known)
- Discussed critical value and p-value approaches to hypothesis testing
- Performed one-tail and two-tail tests
- Performed t test for the mean (σ unknown)
- Performed Z test for the proportion
- Discussed type II error and power of the test
- Performed a hypothesis test for the variance (χ^2)