

Econ 311: Statistical Methods and Interpretation



Chapter 4

Discrete Random Variables and Probability Distributions



Chapter Goals

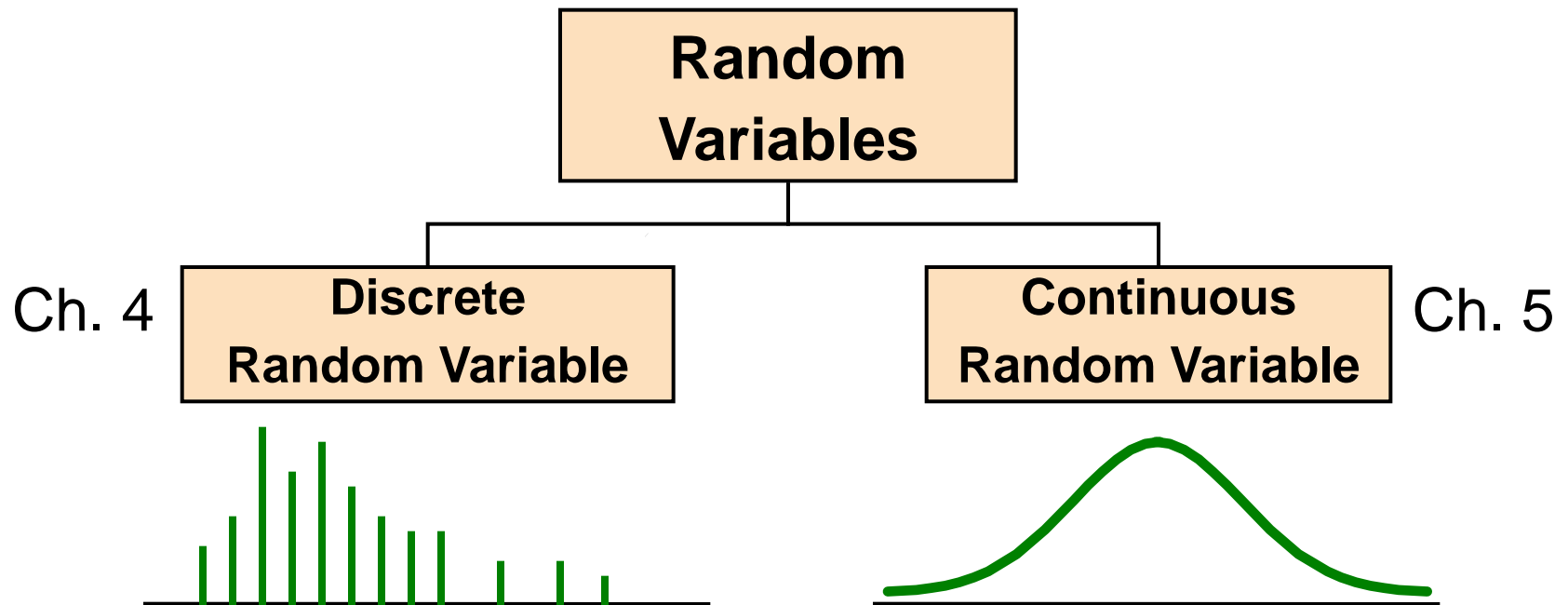
After completing this chapter, you should be able to:

- Interpret the mean and standard deviation for a discrete random variable
- Use the binomial probability distribution to find probabilities
- Describe when to apply the binomial distribution
- Use the hypergeometric and Poisson discrete probability distributions to find probabilities
- Explain covariance and correlation for jointly distributed discrete random variables

Introduction to Probability Distributions

■ Random Variable

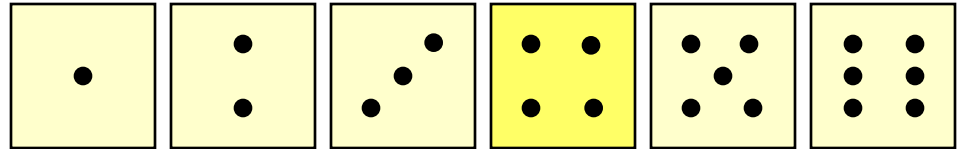
- Represents a possible numerical value from a random experiment



Discrete Random Variables

- Can only take on a countable number of values

Examples:



- **Roll a die twice**

**Let X be the number of times 4 comes up
(then X could be 0, 1, or 2 times)**

- **Toss a coin 5 times.**

**Let X be the number of heads
(then $X = 0, 1, 2, 3, 4, \text{ or } 5$)**

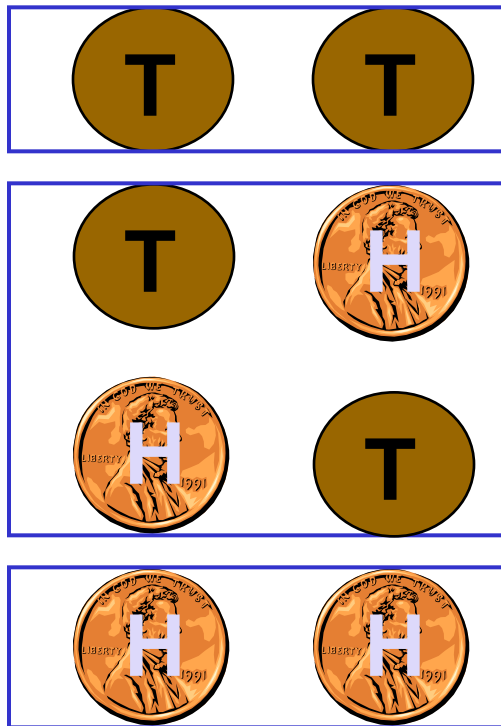


Discrete Probability Distribution

Experiment: Toss 2 Coins. Let $X = \#$ heads.

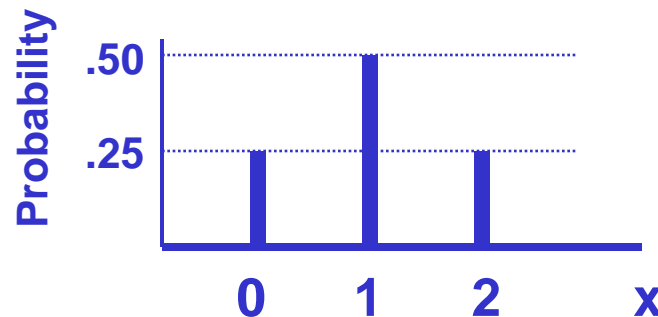
Show $P(x)$, i.e., $P(X = x)$, for all values of x :

4 possible outcomes



Probability Distribution

| <u>x Value</u> | <u>Probability</u> |
|----------------|--------------------|
| 0 | $1/4 = .25$ |
| 1 | $2/4 = .50$ |
| 2 | $1/4 = .25$ |



Probability Distribution Required Properties

- $P(x) \geq 0$ for any value of x
- The individual probabilities **sum to 1**;

$$\sum_x P(x) = 1$$

(The notation indicates summation over all possible x values)



Cumulative Probability Function

- The **cumulative probability function**, denoted $F(x_0)$, shows the probability that X is less than or equal to x_0

$$F(x_0) = P(X \leq x_0)$$

- In other words,

$$F(x_0) = \sum_{x \leq x_0} P(x)$$



Expected Value

- **Expected Value (or mean)** of a discrete distribution (Weighted Average)

$$\mu = E(X) = \sum_x xP(x)$$

- **Example:** Toss 2 coins,
 $x = \#$ of heads,
compute expected value of x :

$$E(x) = (0 \times .25) + (1 \times .50) + (2 \times .25) \\ = 1.0$$

| x | P(x) |
|---|------|
| 0 | .25 |
| 1 | .50 |
| 2 | .25 |



Variance and Standard Deviation

- **Variance** of a discrete random variable X

$$\sigma^2 = E(X - \mu)^2 = \sum_x (x - \mu)^2 P(x)$$

- **Standard Deviation** of a discrete random variable X

$$\sigma = \sqrt{\sigma^2} = \sqrt{\sum_x (x - \mu)^2 P(x)}$$



Standard Deviation Example

- **Example:** Toss 2 coins, $X = \#$ heads, compute standard deviation (recall $E(x) = 1$)

$$\sigma = \sqrt{\sum_x (x - \mu)^2 P(x)}$$

$$\sigma = \sqrt{(0-1)^2(.25) + (1-1)^2(.50) + (2-1)^2(.25)} = \sqrt{.50} = .707$$

Possible number of heads
= 0, 1, or 2



Functions of Random Variables

- If $P(x)$ is the probability function of a discrete random variable X , and $g(X)$ is some function of X , then the expected value of function g is

$$E[g(X)] = \sum_x g(x)P(x)$$



Linear Functions of Random Variables

- Let a and b be any constants.

- a) $E(a) = a$ and $Var(a) = 0$

i.e., if a random variable always takes the value a ,
it will have mean a and variance 0

- b) $E(bX) = b\mu_x$ and $Var(bX) = b^2\sigma_x^2$

i.e., the expected value of $b \cdot X$ is $b \cdot E(x)$



Linear Functions of Random Variables

(continued)

- Let random variable X have mean μ_x and variance σ_x^2
- Let a and b be any constants.
- Let $Y = a + bX$
- Then the mean and variance of Y are

$$\mu_Y = E(a + bX) = a + b\mu_X$$

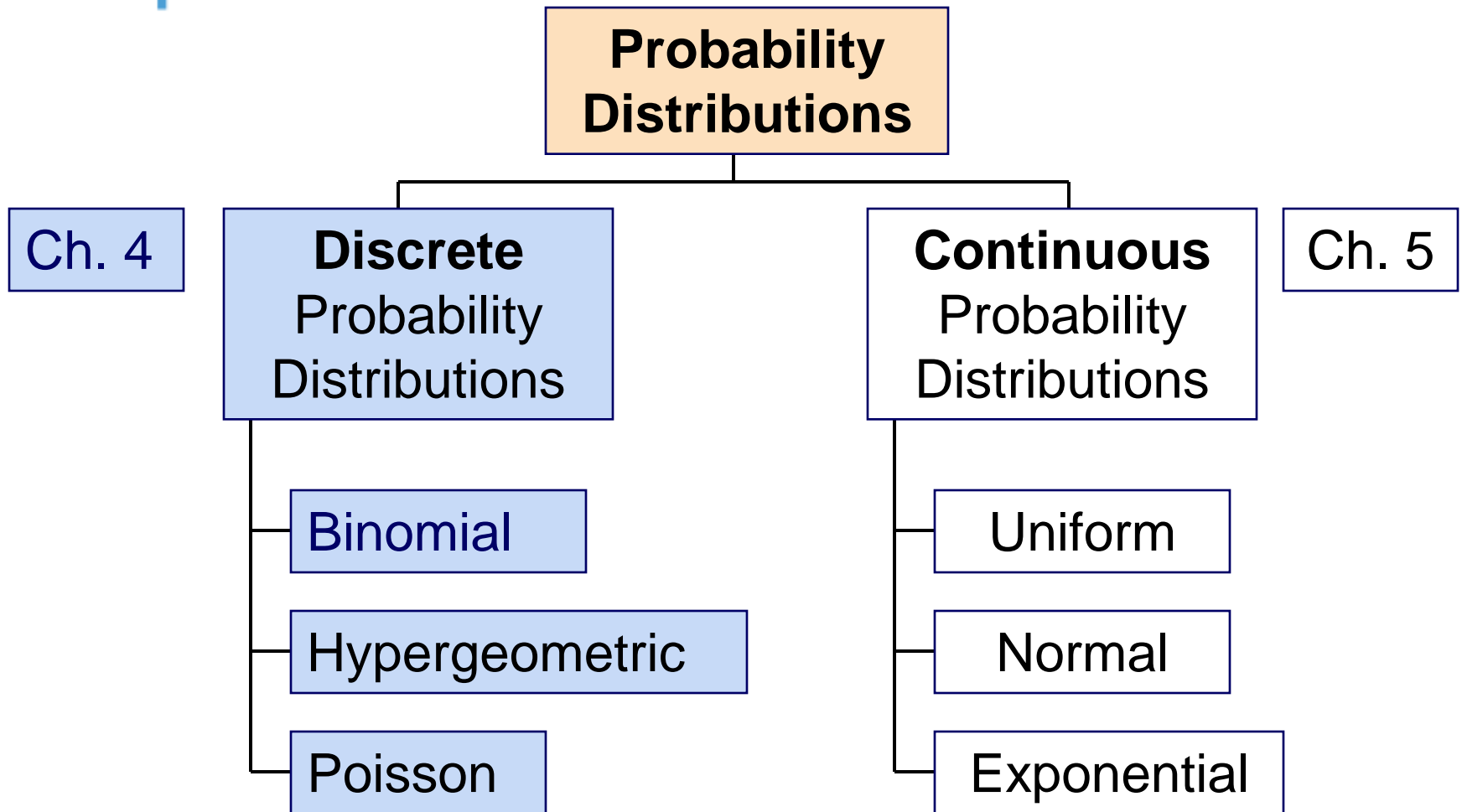
$$\sigma_Y^2 = \text{Var}(a + bX) = b^2\sigma_X^2$$

- so that the standard deviation of Y is

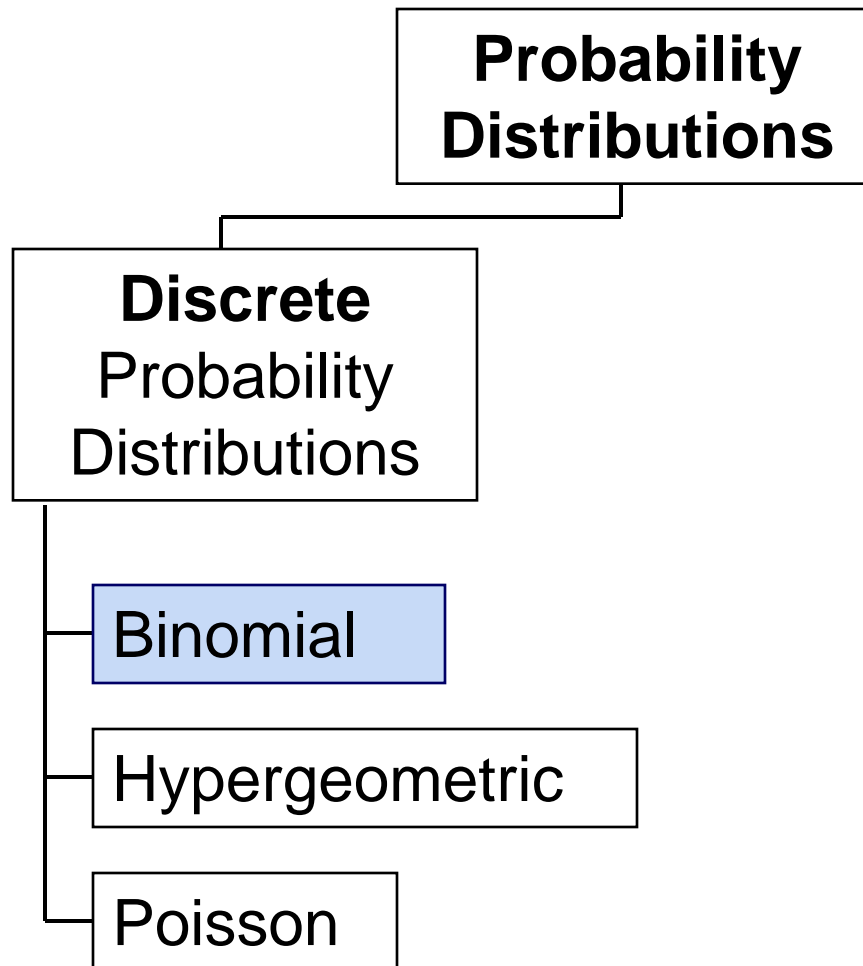
$$\sigma_Y = |b|\sigma_X$$



Probability Distributions



The Binomial Distribution





Bernoulli Distribution

- Consider only two outcomes: “**success**” or “**failure**”
- Let P denote the probability of success
- Let $1 - P$ be the probability of failure
- Define random variable X :
$$x = 1 \text{ if success, } x = 0 \text{ if failure}$$
- Then the **Bernoulli probability function** is

$$P(0) = (1 - P) \quad \text{and} \quad P(1) = P$$



Bernoulli Distribution Mean and Variance

- The mean is $\mu = P$

$$\mu = E(X) = \sum_x xP(x) = (0)(1-P) + (1)P = P$$

- The variance is $\sigma^2 = P(1 - P)$

$$\begin{aligned}\sigma^2 &= E[(X - \mu)^2] = \sum_x (x - \mu)^2 P(x) \\ &= (0 - P)^2 (1 - P) + (1 - P)^2 P = P(1 - P)\end{aligned}$$



Sequences of x Successes in n Trials

- The number of sequences with x successes in n independent trials is:

$$C_x^n = \frac{n!}{x!(n-x)!}$$

Where $n! = n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot 1$ and $0! = 1$

- These sequences are mutually exclusive, since no two can occur at the same time



Binomial Probability Distribution

- A fixed number of observations, n
 - e.g., 15 tosses of a coin; ten light bulbs taken from a warehouse
- Two mutually exclusive and collectively exhaustive categories
 - e.g., head or tail in each toss of a coin; defective or not defective light bulb
 - Generally called “success” and “failure”
 - Probability of success is P , probability of failure is $1 - P$
- Constant probability for each observation
 - e.g., Probability of getting a tail is the same each time we toss the coin
- Observations are independent
 - The outcome of one observation does not affect the outcome of the other



Possible Binomial Distribution Settings

- A manufacturing plant labels items as either defective or acceptable
- A firm bidding for contracts will either get a contract or not
- A marketing research firm receives survey responses of “yes I will buy” or “no I will not”
- New job applicants either accept the offer or reject it



Binomial Distribution Formula

$$P(x) = \frac{n!}{x! (n - x)!} P^x (1 - P)^{n - x}$$

$P(x)$ = probability of x successes in n trials,
with probability of success P on each trial

x = number of 'successes' in sample,
($x = 0, 1, 2, \dots, n$)

n = sample size (number of trials
or observations)

P = probability of "success"

Example: Flip a coin four
times, let $x = \#$ heads:

$$n = 4$$

$$P = 0.5$$

$$1 - P = (1 - 0.5) = 0.5$$

$$x = 0, 1, 2, 3, 4$$



Example: Calculating a Binomial Probability

What is the probability of one success in five observations if the probability of success is 0.1?

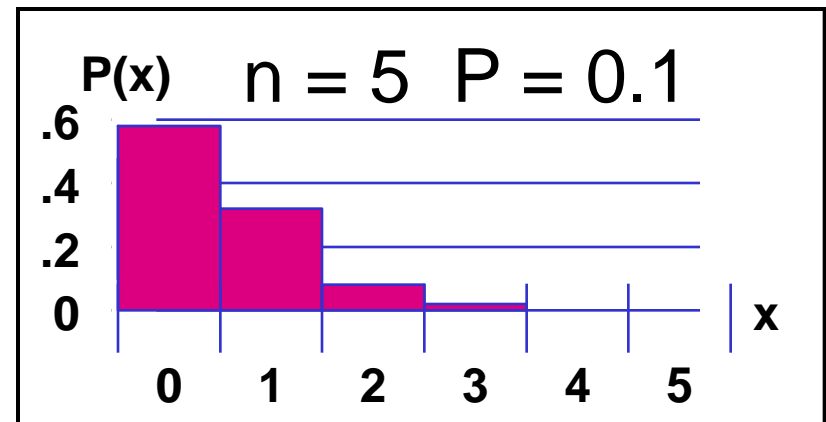
$$x = 1, n = 5, \text{ and } P = 0.1$$

$$\begin{aligned} P(x = 1) &= \frac{n!}{x!(n-x)!} P^x (1-P)^{n-x} \\ &= \frac{5!}{1!(5-1)!} (0.1)^1 (1-0.1)^{5-1} \\ &= (5)(0.1)(0.9)^4 \\ &= .32805 \end{aligned}$$

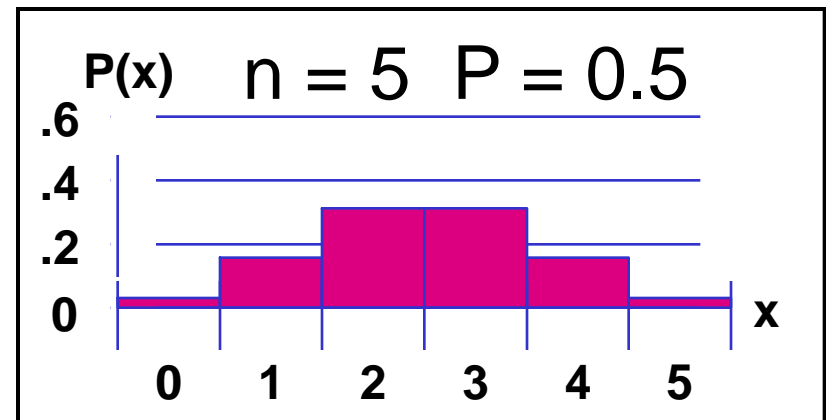
Binomial Distribution

- The shape of the binomial distribution depends on the values of P and n

- Here, $n = 5$ and $P = 0.1$



- Here, $n = 5$ and $P = 0.5$





Binomial Distribution Mean and Variance

- Mean

$$\mu = E(x) = nP$$

- Variance and Standard Deviation

$$\sigma^2 = nP(1-P)$$

$$\sigma = \sqrt{nP(1-P)}$$

Where n = sample size

P = probability of success

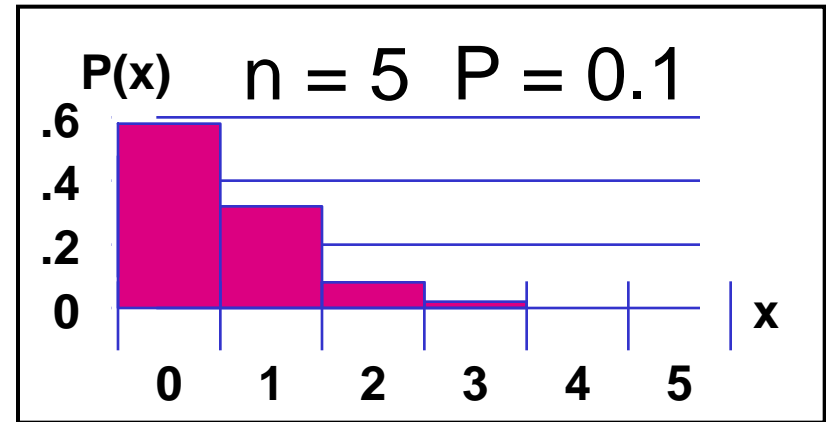
$(1 - P)$ = probability of failure

Binomial Characteristics

Examples

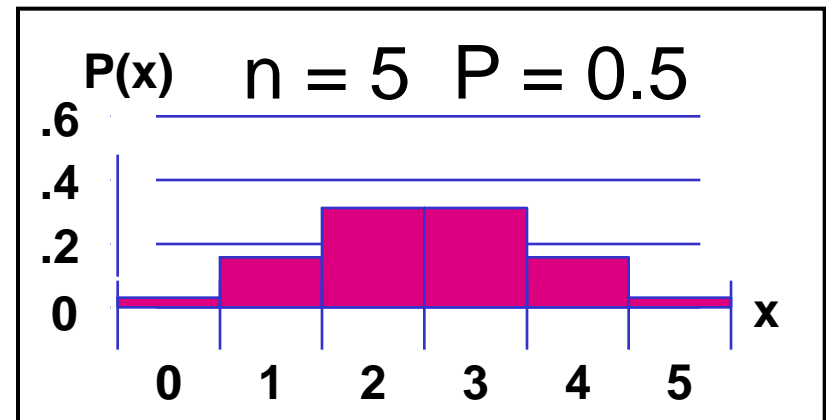
$$\mu = nP = (5)(0.1) = 0.5$$

$$\sigma = \sqrt{nP(1-P)} = \sqrt{(5)(0.1)(1-0.1)} = 0.6708$$



$$\mu = nP = (5)(0.5) = 2.5$$

$$\sigma = \sqrt{nP(1-P)} = \sqrt{(5)(0.5)(1-0.5)} = 1.118$$



Using Binomial Tables

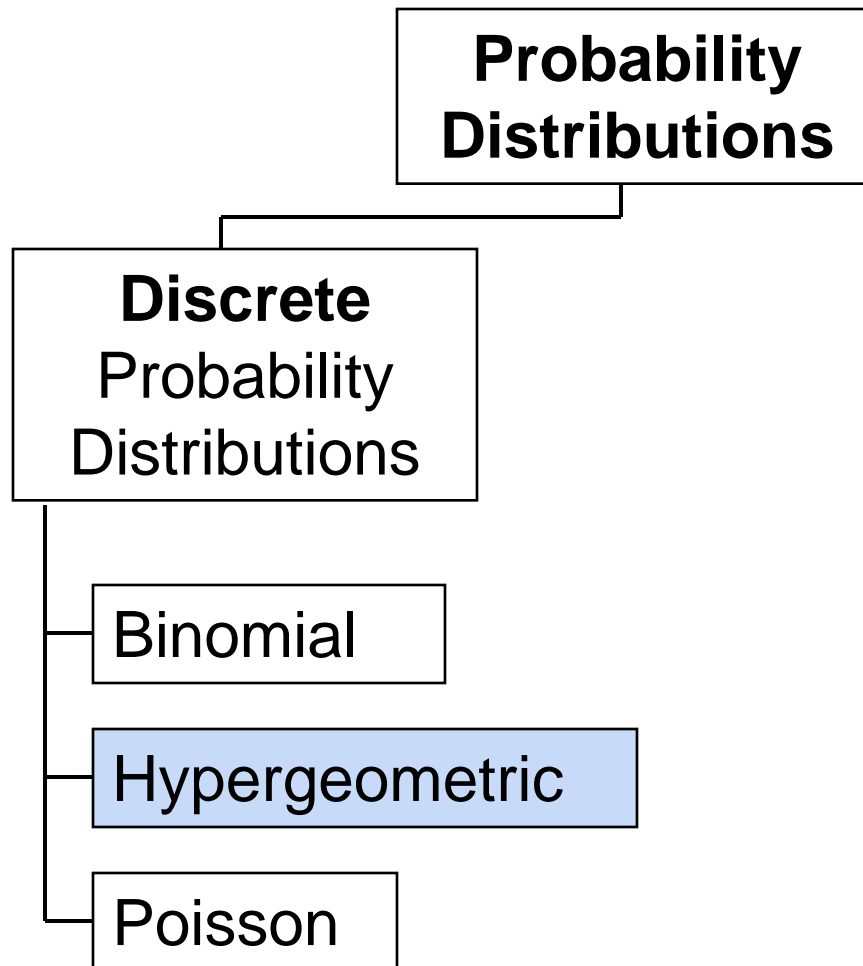
| N | x | ... | p=.20 | p=.25 | p=.30 | p=.35 | p=.40 | p=.45 | p=.50 |
|----|----|-----|--------|--------|--------|---------------|--------|---------------|--------|
| 10 | 0 | ... | 0.1074 | 0.0563 | 0.0282 | 0.0135 | 0.0060 | 0.0025 | 0.0010 |
| | 1 | ... | 0.2684 | 0.1877 | 0.1211 | 0.0725 | 0.0403 | 0.0207 | 0.0098 |
| | 2 | ... | 0.3020 | 0.2816 | 0.2335 | 0.1757 | 0.1209 | 0.0763 | 0.0439 |
| | 3 | ... | 0.2013 | 0.2503 | 0.2668 | 0.2522 | 0.2150 | 0.1665 | 0.1172 |
| | 4 | ... | 0.0881 | 0.1460 | 0.2001 | 0.2377 | 0.2508 | 0.2384 | 0.2051 |
| | 5 | ... | 0.0264 | 0.0584 | 0.1029 | 0.1536 | 0.2007 | 0.2340 | 0.2461 |
| | 6 | ... | 0.0055 | 0.0162 | 0.0368 | 0.0689 | 0.1115 | 0.1596 | 0.2051 |
| | 7 | ... | 0.0008 | 0.0031 | 0.0090 | 0.0212 | 0.0425 | 0.0746 | 0.1172 |
| | 8 | ... | 0.0001 | 0.0004 | 0.0014 | 0.0043 | 0.0106 | 0.0229 | 0.0439 |
| | 9 | ... | 0.0000 | 0.0000 | 0.0001 | 0.0005 | 0.0016 | 0.0042 | 0.0098 |
| | 10 | ... | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0003 | 0.0010 |

Examples:

$$n = 10, x = 3, P = 0.35: \quad P(x = 3|n = 10, p = 0.35) = .2522$$

$$n = 10, x = 8, P = 0.45: \quad P(x = 8|n = 10, p = 0.45) = .0229$$

The Hypergeometric Distribution





The Hypergeometric Distribution

- “n” trials in a sample taken from a **finite population** of size N
- Sample taken **without replacement**
- Outcomes of trials are **dependent**
- Concerned with finding the probability of “X” successes in the sample where there are “S” successes in the population



Hypergeometric Distribution Formula

$$P(x) = \frac{C_x^S C_{n-x}^{N-S}}{C_n^N} = \frac{S!}{x!(S-x)!} \times \frac{(N-S)!}{(n-x)!(N-S-n+x)!} \times \frac{N!}{n!(N-n)!}$$

Where

N = population size

S = number of successes in the population

$N - S$ = number of failures in the population

n = sample size

x = number of successes in the sample

$n - x$ = number of failures in the sample



Using the Hypergeometric Distribution

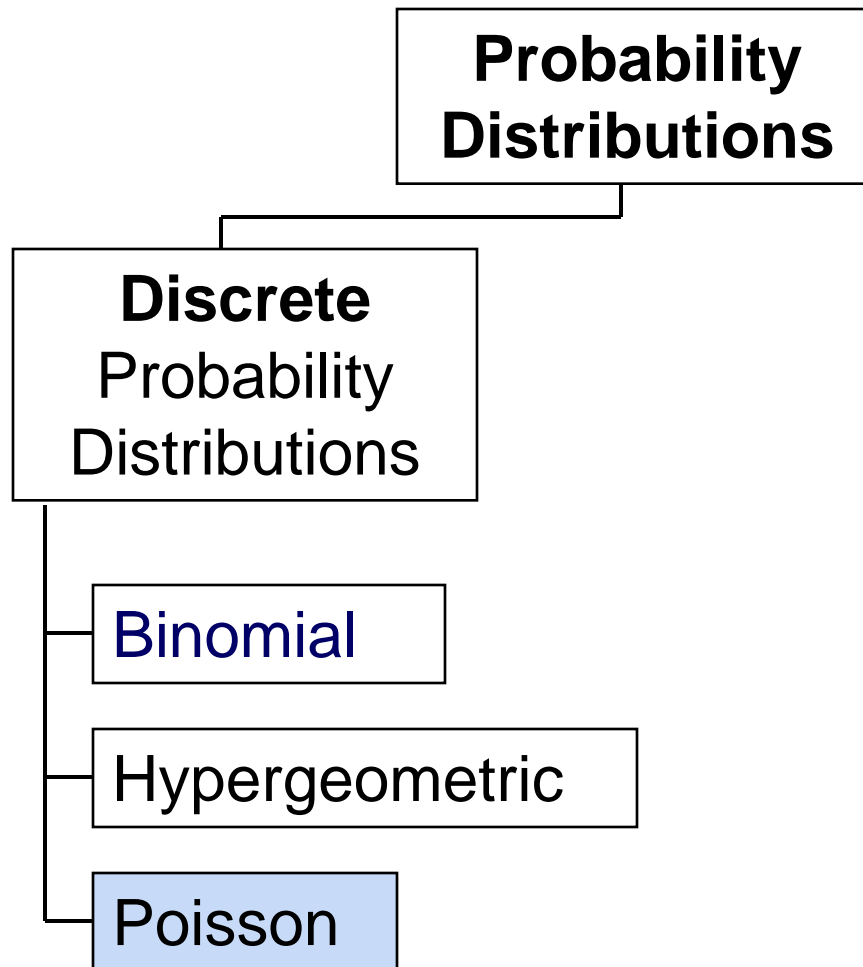
- **Example:** 3 different computers are checked from 10 in the department. 4 of the 10 computers have illegal software loaded. What is the probability that 2 of the 3 selected computers have illegal software loaded?

| | |
|----------|---------|
| $N = 10$ | $n = 3$ |
| $S = 4$ | $x = 2$ |

$$P(x = 2) = \frac{C_x^S C_{n-x}^{N-S}}{C_n^N} = \frac{C_2^4 C_1^6}{C_3^{10}} = \frac{(6)(6)}{120} = 0.3$$

The probability that 2 of the 3 selected computers have illegal software loaded is 0.30, or 30%.

The Poisson Distribution





The Poisson Distribution

- Apply the Poisson Distribution when:
 - You wish to count the number of times an event occurs in a given continuous interval
 - The probability that an event occurs in one subinterval is very small and is the same for all subintervals
 - The number of events that occur in one subinterval is independent of the number of events that occur in the other subintervals
 - There can be no more than one occurrence in each subinterval
 - The **expected number of events per unit** is λ (lambda)



Poisson Distribution Formula

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

where:

x = number of successes per unit

λ = expected number of successes per unit

e = base of the natural logarithm system (2.71828...)



Poisson Distribution Characteristics

- Mean

$$\mu = E(X) = \lambda$$

- Variance and Standard Deviation

$$\sigma^2 = E[(X - \mu)^2] = \lambda$$

$$\sigma = \sqrt{\lambda}$$

where λ = expected number of successes per unit

Using Poisson Tables

| x | λ | | | | | | | | |
|---|-----------|--------|--------|--------|---------------|--------|--------|--------|--------|
| | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 |
| 0 | 0.9048 | 0.8187 | 0.7408 | 0.6703 | 0.6065 | 0.5488 | 0.4966 | 0.4493 | 0.4066 |
| 1 | 0.0905 | 0.1637 | 0.2222 | 0.2681 | 0.3033 | 0.3293 | 0.3476 | 0.3595 | 0.3659 |
| 2 | 0.0045 | 0.0164 | 0.0333 | 0.0536 | 0.0758 | 0.0988 | 0.1217 | 0.1438 | 0.1647 |
| 3 | 0.0002 | 0.0011 | 0.0033 | 0.0072 | 0.0126 | 0.0198 | 0.0284 | 0.0383 | 0.0494 |
| 4 | 0.0000 | 0.0001 | 0.0003 | 0.0007 | 0.0016 | 0.0030 | 0.0050 | 0.0077 | 0.0111 |
| 5 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0002 | 0.0004 | 0.0007 | 0.0012 | 0.0020 |
| 6 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0002 | 0.0003 |
| 7 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

Example: Find $P(X = 2)$ if $\lambda = .50$

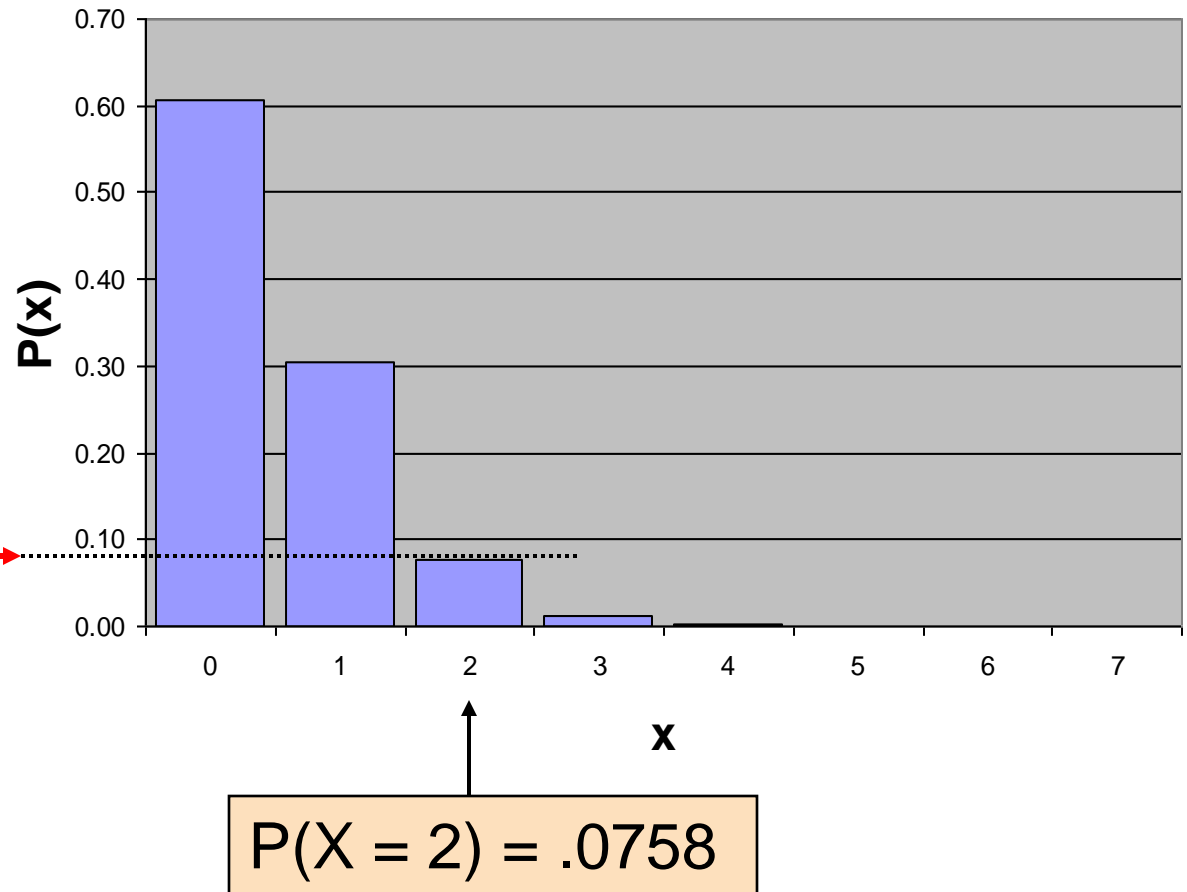
$$P(X = 2) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-0.50} (0.50)^2}{2!} = .0758$$

Graph of Poisson Probabilities

Graphically:

$\lambda = .50$

| X | $\lambda =$ 0.50 |
|----------|--|
| 0 | 0.6065 |
| 1 | 0.3033 |
| 2 | 0.0758 |
| 3 | 0.0126 |
| 4 | 0.0016 |
| 5 | 0.0002 |
| 6 | 0.0000 |
| 7 | 0.0000 |

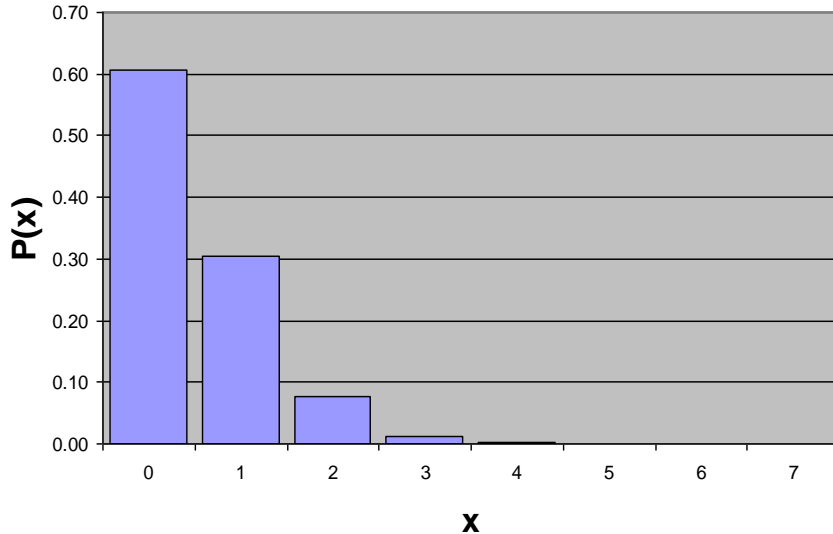




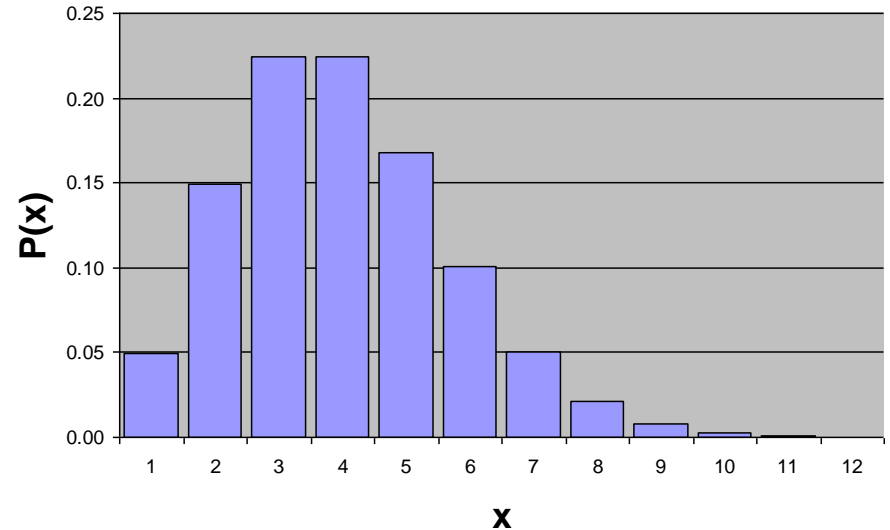
Poisson Distribution Shape

- The shape of the Poisson Distribution depends on the parameter λ :

$\lambda = 0.50$



$\lambda = 3.00$



Joint Probability Functions

- A **joint probability function** is used to express the probability that X takes the specific value x and simultaneously Y takes the value y , as a function of x and y

$$P(x, y) = P(X = x \cap Y = y)$$

- The marginal probabilities are

$$P(x) = \sum_y P(x, y)$$

$$P(y) = \sum_x P(x, y)$$



Conditional Probability Functions

- The **conditional probability function** of the random variable Y expresses the probability that Y takes the value y when the value x is specified for X .

$$P(y | x) = \frac{P(x, y)}{P(x)}$$

- Similarly, the conditional probability function of X , given $Y = y$ is:

$$P(x | y) = \frac{P(x, y)}{P(y)}$$



Independence

- The jointly distributed random variables X and Y are said to be **independent** if and only if their joint probability function is the product of their marginal probability functions:

$$P(x, y) = P(x)P(y)$$

for all possible pairs of values x and y

- A set of k random variables are independent if and only if

$$P(x_1, x_2, \dots, x_k) = P(x_1)P(x_2) \cdots P(x_k)$$



Conditional Mean and Variance

- The conditional mean is

$$\mu_{Y|X} = E[Y | X] = \sum_Y (y | x) P(y | x)$$

- The conditional variance is

$$\sigma_{Y|X}^2 = E[(Y - \mu_{Y|X})^2 | X] = \sum_Y [(y - \mu_{Y|X})^2 | x] P(y | x)$$



Covariance

- Let X and Y be discrete random variables with means μ_X and μ_Y
- The expected value of $(X - \mu_X)(Y - \mu_Y)$ is called the **covariance** between X and Y
- For discrete random variables

$$\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = \sum_x \sum_y (x - \mu_x)(y - \mu_y)P(x, y)$$

- An equivalent expression is

$$\text{Cov}(X, Y) = E(XY) - \mu_x \mu_y = \sum_x \sum_y xyP(x, y) - \mu_x \mu_y$$



Covariance and Independence

- The covariance measures the strength of the linear relationship between two variables
- If two random variables are statistically independent, the covariance between them is 0
 - The converse is not necessarily true



Correlation

- The **correlation** between X and Y is:

$$\rho = \text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

- $\rho = 0 \Rightarrow$ no linear relationship between X and Y
- $\rho > 0 \Rightarrow$ positive linear relationship between X and Y
 - when X is high (low) then Y is likely to be high (low)
 - $\rho = +1 \Rightarrow$ perfect positive linear dependency
- $\rho < 0 \Rightarrow$ negative linear relationship between X and Y
 - when X is high (low) then Y is likely to be low (high)
 - $\rho = -1 \Rightarrow$ perfect negative linear dependency



Portfolio Analysis

- Let random variable X be the price for stock A
- Let random variable Y be the price for stock B
- The **market value**, W , for the portfolio is given by the linear function

$$W = aX + bY$$

(a is the number of shares of stock A,
 b is the number of shares of stock B)



Portfolio Analysis

(continued)

- The **mean** value for W is

$$\begin{aligned}\mu_W &= E[W] = E[aX + bY] \\ &= a\mu_X + b\mu_Y\end{aligned}$$

- The **variance** for W is

$$\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\text{Cov}(X, Y)$$

or using the correlation formula

$$\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\text{Corr}(X, Y)\sigma_X\sigma_Y$$



Example: Investment Returns

Return per \$1,000 for two types of investments

| $P(x_i, y_i)$ | Economic condition | Investment | |
|---------------|--------------------|----------------|-------------------|
| | | Passive Fund X | Aggressive Fund Y |
| .2 | Recession | - \$ 25 | - \$200 |
| .5 | Stable Economy | + 50 | + 60 |
| .3 | Expanding Economy | + 100 | + 350 |

$$E(x) = \mu_x = (-25)(.2) + (50)(.5) + (100)(.3) = 50$$

$$E(y) = \mu_y = (-200)(.2) + (60)(.5) + (350)(.3) = 95$$

Computing the Standard Deviation for Investment Returns

| P(x _i y _i) | Economic condition | Investment | |
|-----------------------------------|--------------------|----------------|-------------------|
| | | Passive Fund X | Aggressive Fund Y |
| 0.2 | Recession | - \$ 25 | - \$200 |
| 0.5 | Stable Economy | + 50 | + 60 |
| 0.3 | Expanding Economy | + 100 | + 350 |

$$\sigma_x = \sqrt{(-25 - 50)^2(0.2) + (50 - 50)^2(0.5) + (100 - 50)^2(0.3)}$$

$$= 43.30$$

$$\sigma_y = \sqrt{(-200 - 95)^2(0.2) + (60 - 95)^2(0.5) + (350 - 95)^2(0.3)}$$

$$= 193.71$$

Covariance for Investment Returns

| P(x _i y _i) | Economic condition | Investment | |
|-----------------------------------|--------------------|----------------|-------------------|
| | | Passive Fund X | Aggressive Fund Y |
| .2 | Recession | - \$ 25 | - \$200 |
| .5 | Stable Economy | + 50 | + 60 |
| .3 | Expanding Economy | + 100 | + 350 |

$$\begin{aligned}
 \text{Cov}(X, Y) &= (-25 - 50)(-200 - 95)(.2) + (50 - 50)(60 - 95)(.5) \\
 &\quad + (100 - 50)(350 - 95)(.3) \\
 &= 8250
 \end{aligned}$$



Portfolio Example

| | | |
|---------------|----------------------|---------------------|
| Investment X: | $\mu_x = 50$ | $\sigma_x = 43.30$ |
| Investment Y: | $\mu_y = 95$ | $\sigma_y = 193.21$ |
| | $\sigma_{xy} = 8250$ | |

Suppose 40% of the portfolio (P) is in Investment X and 60% is in Investment Y:

$$E(P) = .4(50) + (.6)(95) = 77$$

$$\begin{aligned}\sigma_P &= \sqrt{(.4)^2(43.30)^2 + (.6)^2(193.21)^2 + 2(.4)(.6)(8250)} \\ &= 133.04\end{aligned}$$

The portfolio return and portfolio variability are between the values for investments X and Y considered individually



Interpreting the Results for Investment Returns

- The aggressive fund has a higher expected return, but much more risk

$$\begin{aligned}\mu_y &= 95 > \mu_x = 50 \\ &\text{but} \\ \sigma_y &= 193.21 > \sigma_x = 43.30\end{aligned}$$

- The Covariance of 8250 indicates that the two investments are positively related and will vary in the same direction



Chapter Summary

- Defined discrete random variables and probability distributions
- Discussed the Binomial distribution
- Discussed the Hypergeometric distribution
- Reviewed the Poisson distribution
- Defined covariance and the correlation between two random variables
- Examined application to portfolio investment