

Technical Notes for: Threshold-Based Mixed Bundling

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1 Consumers' Demand

In this section we derive the consumers' demand. Consumers take the deal if their utility is weakly improving compared to choices without the deal. Therefore, we must first derive the demand when deal is not offered. There is a continuum of buyers, distinguished by hunger level $h \in [h_0, \infty)$, with cdf $F(h)$, pdf $f(h)$, and thirst level $t \in \{B, NB\}$.

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1.1 Demand with no deal (linear pricing)

A buyer with hunger level h and thirst level t , takes the prices p_x, p_y as given, and solves:

$$\max_{\substack{0 \leq x \leq \frac{I - p_y y}{p_x} \\ y \in \{0,1\}}} v(x|h) + d(y|t) + \underbrace{I - p_x x - p_y y}_m$$

When the solution to this problem is interior, the demand without the deal is implicitly given by:

$$\begin{aligned} v_x(x^*|h) - p_x &= 0 \\ \text{or } x^* &= v_x^{-1}(p_x|h) \\ \text{and } y^* &= 1_{d_t \geq p_y} \end{aligned}$$

where $d_t = d(1|t) - d(0, t)$ is the benefit from having a drink for a buyer with thirst level $t \in \{B, NB\}$.

There is a possibility of a corner solution in this model. For buyers with $d_t \geq p_y$, there are two possible corner solutions. The first corner solution occurs when the buyer spends all the income on food and drink (no composite good is left, $m = 0$). The demand of these buyers is:

$$\begin{aligned} x^* &= \frac{I - p_y}{p_x} \\ y^* &= 1 \end{aligned}$$

The minimal hunger level, h_1 , beyond which buyers with $d_t \geq p_y$ choose this corner, is found by:

$$v_x^{-1}(p_x|h_1) = \frac{I - p_y}{p_x}$$

The second corner occurs when all the income is spent on food only (because the hunger is so high). The demand is then:

$$\begin{aligned} x^* &= \frac{I}{p_x} \\ y^* &= 0 \end{aligned}$$

The minimal hunger level, h_2 , beyond which buyers with $d_t \geq p_y$ choose this corner, is found by:

$$\begin{aligned} v\left(\frac{I - p_y}{p_x}|h_2\right) + d(1|t) &= v\left(\frac{I}{p_x}|h_2\right) + d(0|t) \\ v\left(\frac{I}{p_x}|h_2\right) - v\left(\frac{I - p_y}{p_x}|h_2\right) &= d_t \end{aligned}$$

For buyers with $d_t < p_y$, the only possible corner is:

$$\begin{aligned} x^* &= \frac{I}{p_x} \\ y^* &= 0 \end{aligned}$$

The minimal hunger level, h_3 , beyond which buyers with $d_t < p_y$ choose this corner, is found by:

$$v_x^{-1}(p_x|h_3) = \frac{I}{p_x}$$

Although the demand in this corner is the same as the demand of buyers with $t \geq p_y$ in the second corner, it is likely that $h_2 > h_3$. For buyers with high valuation of drink, a larger hunger level is needed to make them give up the drink. Thus, must we have $h_1 < h_3 < h_2$.

Thus, the demand functions, without the deal, are:

$d_t < p_y$	$d_t \geq p_y$
$x^*(p_x, p_y, I) = \begin{cases} v_x^{-1}(p_x h) & h_0 \leq h < h_3 \\ \frac{I}{p_x} & h_3 \leq h \end{cases}$	$x^*(p_x, p_y, I) = \begin{cases} v_x^{-1}(p_x h) & h_0 \leq h < h_1 \\ \frac{I-p_y}{p_x} & h_1 \leq h < h_2 \\ \frac{I}{p_x} & h_2 \leq h \end{cases}$
$y^*(p_x, p_y, I) = 0$	$y^*(p_x, p_y, I) = \begin{cases} 1 & 0 \leq h < h_2 \\ 0 & h_2 \leq h \end{cases}$

where the threshold hunger levels are solved from:

$$\begin{aligned} [h_1] & : v_x^{-1}(p_x|h_1) = \frac{I-p_y}{p_x} \\ [h_2] & : v\left(\frac{I}{p_x}|h_2\right) - v\left(\frac{I-p_y}{p_x}|h_2\right) = d_t \\ [h_3] & : v_x^{-1}(p_x|h_3) = \frac{I}{p_x} \end{aligned}$$

1.2 Demand with deal (TBMB)

A buyer with hunger level h and thirst level t , takes the prices p_x, p_y , and threshold a as given, and solves:

$$\max_{\substack{0 \leq x \leq \frac{I-p_y y}{p_x} \\ y \in \{0,1\}}} v(x|h) + d(y|t) + \underbrace{I - p_x x - p_y y}_{m} 1_{(x \geq a)}$$

Buyers will "take the deal", i.e. increase their purchases of food to get a free drink if $x^* < a$ and

$$v(a|h) + d(1|t) + I - p_x a \geq v(x^*|h) + d(y^*|t) + I - p_x x - p_y y^*$$

Note that buyers who bought $x^* \geq a$ without the deal, will now get a free drink and will not change their demand for food (because of separability of utility). Therefore, only buyers who otherwise bought less than the threshold, will buy the threshold amount if the additional benefit from the free drink compensates them for the deviation from optimal food consumption. For simplicity of analysis, we assume that the threshold is set within the interior demand for x , i.e. $a \leq \frac{I-p_y}{p_x}$. The distribution of hunger will be such that only a tiny fraction of buyers will be in the corner, so setting the threshold at such high levels is essentially the same as not having the deal at all.

For buyers with $d_t < p_y$, the **minimal** hunger level, \underline{h}_{NB} , beyond which they take the deal, is found by:

$$v(a|\underline{h}_{NB}) + d(1|t) - p_x a \geq v(x^*|\underline{h}_{NB}) + d(0|t) - p_x x$$

Rearranging the above, gives the cost and benefit from taking the deal for a buyer with $d_t < p_y$.

$$\underbrace{v(a|\underline{h}_{NB}) - v(x^*|\underline{h}_{NB}) + d_t}_{\text{benefit}} + \underbrace{p_x(x^* - a)}_{\text{cost}} = 0$$

Notice that the solution will depend on the thirst level t and on the threshold a .

Similarly, for buyers with $d_t \geq p_y$, we find the **minimal** hunger level beyond which they take the deal:

$$v(a|\underline{h}_B) + d(1|t) - p_x a \geq v(x^*|\underline{h}_B) + d(1|t) - p_x x - p_y$$

Rearranging the above, gives the cost and benefit from taking the deal for a buyer with $d_t \geq p_y$.

$$\underbrace{v(a|\underline{h}_B) - v(x^*|\underline{h}_B) + p_y}_{\text{benefit}} + \underbrace{p_x(x^* - a)}_{\text{cost}} = 0$$

For both types of buyers, if a is within the range of interior demand for x , then we can find the **maximal** hunger level, \bar{h} , beyond which buyers' demand for food is unaffected by the deal (i.e. bought $x^* > a$ without the deal):

$$v_x(a|\bar{h}) = p_x$$

Thus, the demand with the deal is:

$d_t < p_y$	$d_t \geq p_y$
$x = \begin{cases} v_x^{-1}(p_x h) & h_0 \leq h < \underline{h}_{NB} \\ a & \underline{h}_{NB} \leq h < \bar{h} \\ v_x^{-1}(p_x h) & \bar{h} \leq h < h_3 \\ \frac{I}{p_x} & h_3 \leq h \end{cases}$	$x = \begin{cases} v_x^{-1}(p_x h) & h_0 \leq h < \underline{h}_B \\ a & \underline{h}_B \leq h < \bar{h} \\ v_x^{-1}(p_x h) & \bar{h} \leq h < h_1 \\ \frac{I-p_y}{p_x} & h_1 \leq h < h_2 \\ \frac{I}{p_x} & h_2 \leq h \end{cases}$
$y = 0$	$y = \begin{cases} 1 & 0 \leq h < \underline{h}_B \\ 0 & \underline{h}_B \leq h \end{cases}$
$y_{f,NB} = 1_{(\underline{h}_{NB} \leq h)}$	$y_{f,B} = 1_{(\underline{h}_B \leq h)}$

1.3 Example: CRRA utility from food

Suppose that utility from food is given by:

$$v(x|h) = \begin{cases} \frac{h x^{1-\rho}-1}{1-\rho} & \rho \geq 0, \rho \neq 1 \\ h \log(x) & \rho = 1 \end{cases}$$

We first find the demand without the deal. At the interior solution, the demand for food is given by:

$$\begin{aligned} h x^{-\rho} &= p_x \\ x &= \left(\frac{h}{p_x}\right)^{1/\rho} \end{aligned}$$

With log utility, the interior demand for food is:

$$x = \frac{h}{p_x}$$

For any amount of food, we can find the hunger parameter that creates this demand as an interior solution:

$$h = p_x x^\rho$$

The hunger level, h_3 , beyond which $d_t < p_y$ buyers choose a corner solution, is given by:

$$h_3 = p_x \left(\frac{I}{p_x} \right)^\rho = I^\rho p_x^{1-\rho}$$

The hunger level, h_1 , beyond which $d_t \geq p_y$ choose the first corner, is found by:

$$h_1 = p_x \left(\frac{I - p_y}{p_x} \right)^\rho$$

The hunger level, h_2 , beyond which $d_t \geq p_y$ choose the second corner, is found by:

$$\begin{aligned} v \left(\frac{I}{p_x} | h_2 \right) - v \left(\frac{I - p_y}{p_x} | h_2 \right) &= d_B \\ h_2 \frac{\left(\frac{I}{p_x} \right)^{1-\rho} - 1}{1 - \rho} - h_2 \frac{\left(\frac{I - p_y}{p_x} \right)^{1-\rho} - 1}{1 - \rho} &= d_B \\ h_2 \left[\frac{I^{1-\rho} - (I - p_y)^{1-\rho}}{p_x^{1-\rho} (1 - \rho)} \right] &= d_B \\ h_2 &= \left[\frac{p_x^{1-\rho} (1 - \rho)}{I^{1-\rho} - (I - p_y)^{1-\rho}} \right] d_B \end{aligned}$$

In the log utility case, this becomes:

$$\begin{aligned} h_2 \log \left(\frac{I}{p_x} \right) - h_2 \log \left(\frac{I - p_y}{p_x} \right) &= d_B \\ h_2 \left[\log \left(\frac{I}{p_x} \right) - \log \left(\frac{I - p_y}{p_x} \right) \right] &= d_B \\ h_2 \left[\log \left(\frac{I}{I - p_y} \right) \right] &= d_B \\ h_2 &= d_B \left[\log \left(\frac{I}{I - p_y} \right) \right]^{-1} \end{aligned}$$

Now, suppose that the deal is introduced, and we wish to identify the buyers who "take

the deal", i.e. finding \underline{h}_{NB} , \underline{h}_B and \bar{h} . First, we find \underline{h}_{NB} by solving:

$$\underbrace{v(a|\underline{h}_{NB}) - v(x^*|\underline{h}_{NB}) + d_t}_{\text{benefit}} + \underbrace{p_x(x^* - a)}_{\text{cost}} = 0$$

$$\underline{h}_{NB} \frac{a^{1-\rho} - 1}{1 - \rho} - \underline{h}_{NB} \frac{\left[\left(\frac{\underline{h}_{NB}}{p_x}\right)^{1/\rho}\right]^{1-\rho} - 1}{1 - \rho} + p_x \left[\left(\frac{\underline{h}_{NB}}{p_x}\right)^{1/\rho} - a \right] + d_{NB} = 0$$

$$\underline{h}_{NB} \frac{a^{1-\rho} - 1}{1 - \rho} - \underline{h}_{NB}^{1/\rho} \frac{p_x^{(\rho-1)/\rho} - 1}{1 - \rho} + \underline{h}_{NB}^{1/\rho} p_x^{(\rho-1)/\rho} - p_x a + d_{NB} = 0$$

This equation can be solved only numerically. In the log utility case, the above becomes:

$$\underline{h}_{NB} \log(a) - \underline{h}_{NB} \log\left(\frac{\underline{h}_{NB}}{p_x}\right) + p_x \left(\frac{\underline{h}_{NB}}{p_x} - a\right) + d_{NB} = 0$$

$$\underline{h}_{NB} \log(a) - \underline{h}_{NB} \log\left(\frac{\underline{h}_{NB}}{p_x}\right) + \underline{h}_{NB} - p_x a + d_{NB} = 0$$

This equation can be solved only numerically.

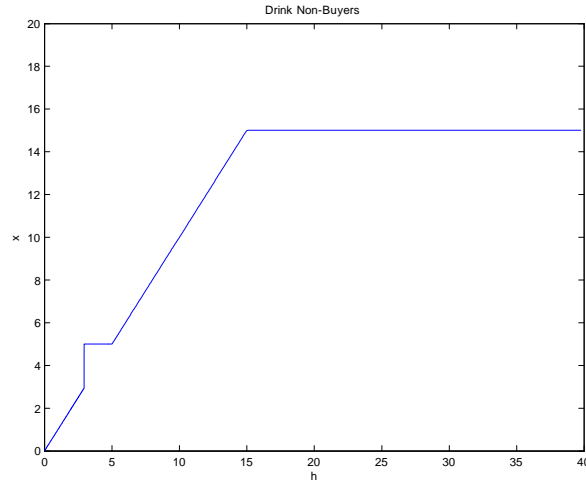
Similarly, we find \underline{h}_B by solving:

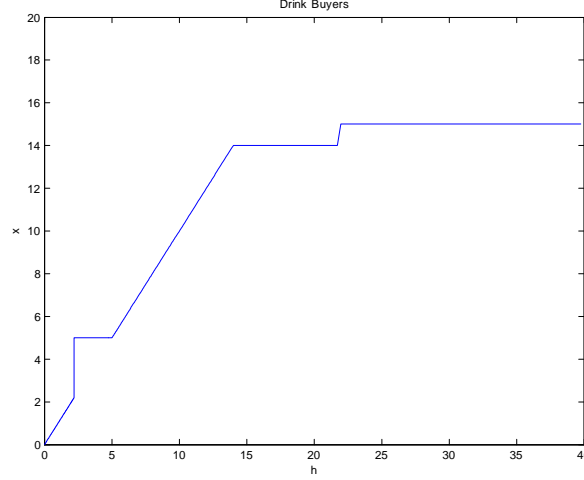
$$\underbrace{v(a|\underline{h}_B) - v(x^*|\underline{h}_B) + p_y}_{\text{benefit}} + \underbrace{p_x(x^* - a)}_{\text{cost}} = 0$$

$$\underline{h}_B \frac{a^{1-\rho} - 1}{1 - \rho} - \underline{h}_B \frac{\left(\frac{\underline{h}_B}{p_x}\right)^{(1-\rho)/\rho} - 1}{1 - \rho} + p_x \left[\left(\frac{\underline{h}_B}{p_x}\right)^{1/\rho} - a \right] + p_y = 0$$

$$\underline{h}_B \frac{a^{1-\rho} - 1}{1 - \rho} - \underline{h}_B^{1/\rho} \frac{p_x^{(\rho-1)/\rho} - 1}{1 - \rho} + \underline{h}_B^{1/\rho} p_x^{(\rho-1)/\rho} - p_x a + p_y = 0$$

The next figure illustrates the demand for food as a function of hunger:





1.4 Aggregate demand with TBMB

Suppose there is a measure μ of buyers with $d_t < p_y$ and a measure $1 - \mu$ of buyers with $d_t \geq p_y$. The aggregate (expected) demand by the buyers with $d_t < p_y$ is given by:

$$\begin{aligned}
 E(x|NB, a) &= \int_{h_0}^{\bar{h}_{NB}} v_x^{-1}(p_x|h) dF(h) + \int_{\bar{h}_{NB}}^{\bar{h}} a dF(h) \\
 &\quad + \int_{\bar{h}}^{h_3} v_x^{-1}(p_x|h) dF(h) + \int_{h_3}^{\infty} \left(\frac{I}{p_x}\right) dF(h)
 \end{aligned}$$

In the CRRA utility case, the expected demand of $d_t < p_y$ buyers is:

$$\begin{aligned}
 E(x|NB, a) &= \int_{h_0}^{\bar{h}_{NB}} \left(\frac{h}{p_x}\right)^{1/\rho} dF(h) + a \int_{\bar{h}_{NB}}^{\bar{h}} dF(h) \\
 &\quad + \int_{\bar{h}}^{h_3} \left(\frac{h}{p_x}\right)^{1/\rho} dF(h) + \left(\frac{I}{p_x}\right) \int_{h_3}^{\infty} dF(h)
 \end{aligned}$$

The aggregate (expected) demand by the buyers with $d_t \geq p_y$ is given by:

$$\begin{aligned}
 E(x|B, a) &= \int_{h_0}^{\bar{h}_B} v_x^{-1}(p_x|h) dF(h) + \int_{\bar{h}_B}^{\bar{h}} a dF(h) \\
 &\quad + \int_{\bar{h}}^{h_1} v_x^{-1}(p_x|h) dF(h) + \int_{h_1}^{h_2} \left(\frac{I - p_y y}{p_x}\right) dF(h) + \int_{h_2}^{\infty} \left(\frac{I}{p_x}\right) dF(h)
 \end{aligned}$$

In the CRRA utility case, the expected demand of $d_t \geq p_y$ buyers is:

$$\begin{aligned}
 E(x|B, a) &= \int_{h_0}^{\bar{h}_B} \left(\frac{h}{p_x}\right)^{1/\rho} dF(h) + a \int_{\bar{h}_B}^{\bar{h}} dF(h) \\
 &\quad + \int_{\bar{h}}^{h_1} \left(\frac{h}{p_x}\right)^{1/\rho} dF(h) + \left(\frac{I - p_y y}{p_x}\right) \int_{h_1}^{h_2} dF(h) + \left(\frac{I}{p_x}\right) \int_{h_2}^{\infty} dF(h)
 \end{aligned}$$

The expected demand for food by both groups of thirst is:

$$E(x|a) = \mu E(x|NB, a) + (1 - \mu) E(x|B, a)$$

Notice that the extra (expected) demand for food generated by the deal, is:

$$\Delta E(x) = \mu \int_{\underline{h}_{NB}}^{\bar{h}} [a - v_x^{-1}(p_x|h)] dF(h) + (1 - \mu) \int_{\underline{h}_B}^{\bar{h}} [a - v_x^{-1}(p_x|h)] dF(h)$$

In the CRRA utility case, the extra demand for food generated by the deal, is:

$$\Delta E(x) = \mu \int_{\underline{h}_{NB}}^{\bar{h}} \left[a - \left(\frac{h}{p_x} \right)^{1/\rho} \right] dF(h) + (1 - \mu) \int_{\underline{h}_B}^{\bar{h}} \left[a - \left(\frac{h}{p_x} \right)^{1/\rho} \right] dF(h)$$

The aggregate demand for drink is generated only by buyers with $d_t \geq p_y$, who do not take the deal.

$$E(y|a) = (1 - \mu) \int_{h_0}^{\underline{h}_B} dF(h)$$

Buyers $d_t < p_y$ who are not hungry enough to take the deal, will neither buy the drink nor get it for free. The rest of the buyers will get a free drink:

$$E(y_f|a) = \mu \int_{\underline{h}_{NB}}^{\infty} dF(h) + (1 - \mu) \int_{\underline{h}_B}^{\infty} dF(h)$$

2 Integration

We assume that distribution of hunger is lognormal, i.e.

$$\ln h \sim N(\mu, \sigma^2)$$

When computing the expected demand, we need to compute integrals of the form:

$$1. \int_a^b f(h) dh, \quad 2. \int_a^b h^{1/\rho} f(h) dh$$

where $f(\cdot)$ is the pdf of lognormal random variable. The first integral is just

$$\begin{aligned} \Pr(a \leq h \leq b) &= \Pr(\ln a \leq \ln h \leq \ln b) \\ &= \Pr\left(\frac{\ln a - \mu}{\sigma} \leq \frac{\ln h - \mu}{\sigma} \leq \frac{\ln b - \mu}{\sigma}\right) \\ &= \Phi\left(\frac{\ln b - \mu}{\sigma}\right) - \Phi\left(\frac{\ln a - \mu}{\sigma}\right) \end{aligned}$$

The last step follows from the fact that

$$\frac{\ln h - \mu}{\sigma} \sim N(0, 1)$$

For computation of the second type of integral, notice that $h^{1/\rho}$ has lognormal distribution. In particular,

$$\ln(h^{1/\rho}) = \frac{1}{\rho} \ln h \sim \frac{1}{\rho} N(\mu, \sigma^2) = N\left(\frac{\mu}{\rho}, \left(\frac{\sigma}{\rho}\right)^2\right)$$

and

$$h^{1/\rho} \sim LN\left(\frac{\mu}{\rho}, \left(\frac{\sigma}{\rho}\right)^2\right)$$

Thus, for the second integral, we will need the following theorem.

Theorem 1 *Suppose $y \sim LN(\mu, \sigma^2)$. Then, for $0 \leq a < b \leq \infty$.*

$$\int_a^b y f(y) dy = E(y) [\Phi(b_0 - \sigma) - \Phi(a_0 - \sigma)]$$

where $\Phi(\cdot)$ is the cdf of the standard normal, $E(y) = \exp(\mu + \sigma^2/2)$ is the non-truncated mean, and

$$a_0 = \frac{\ln a - \mu}{\sigma}, \quad b_0 = \frac{\ln b - \mu}{\sigma}$$

Special cases:

$$\begin{aligned} \int_0^b y f(y) dy &= E(y) \Phi(b_0 - \sigma) \\ \int_a^\infty y f(y) dy &= E(y) [1 - \Phi(a_0 - \sigma)] = E(y) \Phi(-a_0 + \sigma) \end{aligned}$$

Applying this theorem to the demand of buyers with $d_t \geq p_y$, recall that

$$h^{1/\rho} \sim LN\left(\frac{\mu}{\rho}, \left(\frac{\sigma}{\rho}\right)^2\right) \equiv LN(\tilde{\mu}, \tilde{\sigma}^2)$$

and therefore

$$E(h^{1/\rho}) = \exp(\tilde{\mu} + \tilde{\sigma}^2/2)$$

Thus, the first term in the demand can be written as:

$$\begin{aligned} \int_0^{\underline{h}_B} \left(\frac{h}{p_x}\right)^{1/\rho} dF(h) &= \left(\frac{1}{p_x}\right)^{1/\rho} \int_0^{\underline{h}_B} h^{1/\rho} f(h) d(h) \\ &= \left(\frac{1}{p_x}\right)^{1/\rho} E(h^{1/\rho}) \Phi(b_0 - \tilde{\sigma}) \end{aligned}$$

where

$$b_0 = \frac{\ln \underline{h}_B - \tilde{\mu}}{\tilde{\sigma}}$$

Similarly, the third term in the demand is:

$$\begin{aligned} \int_{\bar{h}}^{h_1} \left(\frac{h}{p_x} \right)^{1/\rho} dF(h) &= \left(\frac{1}{p_x} \right)^{1/\rho} \int_{\bar{h}}^{h_1} h^{1/\rho} dF(h) \\ &= \left(\frac{1}{p_x} \right)^{1/\rho} E(h^{1/\rho}) [\Phi(b_0 - \tilde{\sigma}) - \Phi(a_0 - \tilde{\sigma})] \end{aligned}$$

where

$$b_0 = \frac{\ln h_1 - \tilde{\mu}}{\tilde{\sigma}}, \quad a_0 = \frac{\ln \bar{h} - \tilde{\mu}}{\tilde{\sigma}}$$

The other integrals in the expected demand, do not have the $h^{1/\rho}$ terms, and are computed as follows:

$$\begin{aligned} \int_{h_B}^{\bar{h}} dF(h) &= \Phi\left(\frac{\ln \bar{h} - \mu}{\sigma}\right) - \Phi\left(\frac{\ln h_B - \mu}{\sigma}\right) \\ \int_{h_1}^{h_2} dF(h) &= \Phi\left(\frac{\ln h_2 - \mu}{\sigma}\right) - \Phi\left(\frac{\ln h_1 - \mu}{\sigma}\right) \\ \int_{h_2}^{\infty} dF(h) &= 1 - \Phi\left(\frac{\ln h_2 - \mu}{\sigma}\right) \end{aligned}$$

3 Profit Maximization

3.1 Seller is a Price Taker

The optimization problem of a seller who takes p_x and p_y as given, is:

$$\max_a E(\pi|a) = (p_x - c_x) E(x|a) + (p_y - c_y) E(y|a) - c_y E(y_f|a)$$

Profit with no deal:

$$\pi^* = (p_x - c_x) E(x^*) + (p_y - c_y) E(y^*)$$

3.2 Monopoly

The optimization problem of the seller is:

$$\max_{p_x, p_y, a} E(\pi|\text{deal}) = (p_x - c_x) E(x|a) + (p_y - c_y) E(y|a) - c_y E(y_f|a)$$

Profit with no deal:

$$\max_{p_x, p_y} E(\pi|\text{no deal}) = (p_x - c_x) E(x^*) + (p_y - c_y) E(y^*)$$

4 Elasticity of Substitution Between x and y

Suppose consumers derive utility from three goods: x (food), y (drink), and m (a composite good). Goods x and m can be purchased in any non-negative quantity. Good y can only be purchased in quantities 1 (a drink) or 0 (no drink). We assume that a consumer's utility function is additively separable in all three goods, i.e. the marginal utility of each good is independent of the other two goods. In addition, consumers can vary by hunger level h and thirst level t :

$$u(x, y, m|h, t) = v(x|h) + d(y|t) + m$$

We assume that $v(x|h)$ is twice continuously differentiable, strictly increasing in x and strictly concave. The only assumption made about $d(y|t)$ is that it is increasing in y .

Although y can be consumed in discrete amounts only, this limitation does not preclude us from calculating the elasticity of substitution between x and y .

$$\begin{aligned} \eta_{x,y} &= \frac{d(y/x)}{dMRS_{x,y}} \cdot \frac{MRS_{x,y}}{y/x} \\ MRS_{x,y} &= \frac{v_x(x|h)}{d_y(y|t)} \\ dMRS_{x,y} &= \frac{d_y(y|t) v_{xx}(x|h) dx - v_x(x|h) d_{yy}(y|t) dy}{d_y(y|t)^2} \\ d(y/x) &= \frac{dy \cdot x - y \cdot dx}{x^2} \\ \eta_{x,y} &= \frac{\left[\frac{dy \cdot x - y \cdot dx}{x^2} \right]}{\left[\frac{d_y(y|t) v_{xx}(x|h) dx - v_x(x|h) d_{yy}(y|t) dy}{d_y(y|t)^2} \right]} \cdot \frac{\left[\frac{v_x(x|h)}{d_y(y|t)} \right]}{\left[\frac{y}{x} \right]} \end{aligned}$$

When $d(y|t)$ is linear in y we have:

$$\begin{aligned} \eta_{x,y} &= \frac{\left[\frac{dy \cdot x - y \cdot dx}{x^2} \frac{x}{y} \right]}{\left[\frac{d_y(y|t) v_{xx}(x|h) dx}{d_y(y|t)^2} \frac{d_y(y|t)}{v_x(x|h)} \right]} \\ &= \frac{\left[\frac{dy \cdot x - y \cdot dx}{xy} \right]}{\left[\frac{v_{xx}(x|h) dx}{v_x(x|h)} \right]} \\ &= \frac{\frac{dx}{x} \left[\frac{dy}{dx} \frac{x}{y} - 1 \right]}{\frac{v_{xx}(x|h) dx}{v_x(x|h)}} \\ &= \frac{\frac{dy}{dx} \frac{x}{y} - 1}{\frac{v_{xx}(x|h)}{v_x(x|h)}} x \end{aligned}$$