Technical Appendix for "On the Time Allocation of Married Couples since 1960".

Michael Bar[*] Oksana Leukhina[†]

October 21, 2010

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[*]Bar: Department of Economics, San Francisco State University, San Francisco, CA 94132;
[†]Leukhina: Department of Economics, University of Washington, Seattle, WA 98195.

1 U.S. Census - Data on Worktime, Earnings, Personal Attributes of Married Couples

We download the 1960, 1970, 1980, 1990, 2000 U.S. Census data from IPUMs. Most of the census questions relevant to this project refer to the previous years, i.e. 1959, 1969, ..., 1999. We keep only married non-farm individuals of ages $[25 - 64]$ whose spouse is present and translate all incomes into 1999 dollars using 12 months averages of seasonally adjusted CPI,

Table 1: Consumer Price Index

1959	1969	1979	1989	1999
29.17	36.68	72.58	123.94	166.58

We then create time series that are natural logs of all income types.

We do not correct for topcoding in 89 and 99 because the topcoded observations are already replaced by state mean or median. Hence, we only correct for 59, 69, 79. Using the mean and SD of the truncated distribution of logs of male annual wage incomes, the level of the topcode, and the assumption of the normality of this distribution, we compute the expected mean in the tail of the male wage distribution. The results are reported below.

Table 2: Correction For Topcoding

year	μ_X, truncated	σ_X, truncated	topcode: a	correction: $E[X\|X > a]$
1959	10.19074	0.6695618	11.86896571	12.09612871
1969	10.48966	0.6622401	12.33302225	12.53615899
1979	10.46222	0.7789073	12.05602852	12.39249818

We then replace the topcoded male annual wage income with $E[X|X > a]$. We then replace the topcoded female annual earnings with a*mean(wage of female)/mean(wage of male) of those individuals whose wage exceeds the mean of male earnings and excluding those with topcoded wage income. We deal with topcoded observations of other incomes in the same manner we deal with female wage income.

Once we correct for topcoding we create a new labor income variable

$$\text{Labor Income} = \text{Wage Income} + \text{Business Income} + \text{Farm Income}$$

and drop individuals with negative labor incomes.

We finally need to deal with intervalled variables. Actual weeks worked last year and usual weekly hours worked last year are available since 1979 only. For 1959 and 1969 we are forced to use intervalled counterparts of these series. The objective is to figure out the right midpoints for each of the intervals. To do so we use 1979 data on actual and intervalled series and compute averages for each interval.

Table 3: Availability of Data on Worktime

	1959	1969	1979	1989	1999
Actual Hours	NA	NA	Available	Available	Available
Intervalled Hours	Available	Available	Available	Available	NA
Actual Weeks	NA	NA	Available	Available	Available
Intervalled Weeks	Available	Available	Available	Available	Available

We get different midpoints for men and women.

We drop people with a mismatch between hours and income, i.e. positive hours but negative incomes or vice versa. We drop people with a mismatch between hours and income, i.e. positive hours but negative incomes or vice versa. We then match husbands and wives. Here we keep the following variables: year, household weight, personal weight, husband's and wife's labor incomes last year, their hours, age, race, education record, number of children ever born and number of children under five at home, and class of work (whether they are self-employed, work for wage, or neither), and weeks worked last year.

The number of observations[1] (couples) that we end up with is given by

Table 4: Original Sample Size

Sample 1: year	# couples
1959	$21,897,992$
1969	$24,218,210$
1979	$34,481,282$
1989	$37,712,472$
1999	$42,328,021$

We drop the no earner couples (both husband and wife work 0 hours and earn 0 income). After this the number of available observation changes as follows:

We then drop the 1F couples. The reason for doing this is as follows. One experiment we perform is changing the joint wage distribution of husbands and wives. To estimate the parameters of this distribution we need to correct for the selection bias using our model. It is impossible to do so if we have the selection bias problem for both, males and females. Hence, we only consider the changes in the patterns of 1M and 2E couples. After we this the number of available observation changes as follows:

[1] IPUMS [?] provides a 1% weighted sample of the total population of each census. We use the household's weights in order to campute the number of households in the total population that our sample represents.

Table 5: Final Sample Size

Sample 2: year	# couples	fraction of Sample 1 couples dropped
1959	$21,449,563$	0.020478088
1969	$23,623,713$	0.02454752
1979	$33,093,874$	0.040236555
1989	$36,280,387$	0.037973777
1999	$40,794,924$	0.036219435

Table 6: Sample Size

Sample 3: year	# couples	fraction of Sample 2 couples dropped
1959	$21,204,617$	0.011419627
1969	$23,248,014$	0.01590347
1979	$32,011,871$	0.032694963
1989	$34,907,676$	0.037836173
1999	$38,854,217$	0.047572267

The only people with a mismatch of hours and weeks are some women in 1959 and 1969, whose hours are zero but number of weeks worked is positive. For each of these years these women are less than 0.4% of the sample. We replace these women's weeks worked with a 0.

2 Integration

In this section we describe the methods we use for computing the moments in our model. Green 2000 [1] points out that "A long-standing challenge in applied econometrics has been to obtain a fast and accurate method of computing cumulative probabilities for the bivariate normal distribution". We admit that for us it has been a challenge indeed. Our task was to find efficient techniques for computing moments of the form $E\left[g\left(X,Y\right)\mid\left(X,Y\right)\in A\right]$, where the set A is some nontrivial set in $\mathbb{R}^2$. We start by describing the difficulties in computing the moments in our models and the trade-offs we faced when choosing the integration method. Then, we describe in detail how one can improve the speed of computation by using some basic properties of the truncated normal distribution and appropriate numerical integration techniques.

The Gauss-Hermite quadrature is designed for integrating smooth function over $(-\infty,\infty)$ with respect to the Gaussian distribution. It achieves great accuracy with very few points. We use the Gauss-Hermite quadrature in the computation of open intervals in the model with home production. Since the thresholds are expensive to compute, but they are smooth functions, the Gauss-Hermite quadrature is best suited for those moments. The Gauss-Legendre is efficient for integrating smooth functions over closed intervals. We implement it in the model with home production for computing conditional probabilities on intervals of husband's income. See Miranda 2002 [3], chapter 5, for more details and codes for computing nodes and weights for various quadrature methods.

It is highly desirable to reduce double integrals to single integrals if possible. The next section describes how we can do that in our model. We use two basic properties of the bivariate normal distribution, summarized in the theorems below.

2.1 Notation

- X, Y are random variables.
- x, y are generic realizations of X and Y respectively.
- $f(x, y)$ is the joint distribution of (X, Y).
- $f_X(x)$ and $f_Y(y)$ are the marginal densities, i.e.,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x, y)\, dy \\ f_Y(y) &= \int_{-\infty}^{\infty} f(x, y)\, dx \end{aligned}$$

- $f(y|x)$ is the conditional density of Y given $X = x$, i.e.,

$$f(y|x) = \frac{f(x, y)}{f_X(x)}$$

  We denote the parameters of conditional distribution by $\mu_{Y|X=x}$ and $\sigma_{Y|X=x}$.

- $f(x|y)$ is the conditional density of X given $Y = y$, i.e.,

$$f(x|y) = \frac{f(x, y)}{f_Y(y)}$$

- $F_X(x)$ and $F_Y(y)$ are the unconditional cumulative density functions (c.d.f.) of X and Y respectively. That is,

$$\begin{aligned} F_X(x) &= \Pr(X \leq x) = \int_{-\infty}^{x} f_X(s)\, ds \\ F_Y(y) &= \Pr(Y \leq y) = \int_{-\infty}^{x} f_Y(s)\, ds \end{aligned}$$

- $F(y|x)$ is the conditional c.d.f. of Y given that $X = x$. i.e.,

$$F(y|x) = \Pr(Y \leq y | X = x) = \int_{-\infty}^{y} f(s|x)\, ds$$

- $F(x|y)$ is the conditional c.d.f. of X given that $Y = y$. i.e.,

$$F(x|y) = \Pr(X \leq x | Y = y) = \int_{-\infty}^{x} f(s|y)\, ds$$

- $\phi(z)$ is the probability density function (p.d.f.) of the standard normal, i.e.,

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

- $\Phi(z)$ is the c.d.f. of the standard normal.

2.2 Theorems Used Below

Theorem 1 *Let* $(X, Y) \sim N\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix}\right)$. *Then*

$$\begin{aligned} (Y|X=x) &\sim N\left[\alpha+\beta x, \sigma_Y^2\left(1-\rho^2\right)\right] \\ \alpha &= \mu_Y - \beta\mu_X \\ \beta &= \frac{\sigma_{XY}}{\sigma_X^2},\ \rho = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} \end{aligned}$$

Theorem 2 ***(Moments of truncated normal distribution[2]).*** *Let* $Y \sim N[\mu, \sigma^2]$. *Then*

$$\begin{aligned} E[Y|a_1 \le Y \le a_2] &= \mu - \sigma\left[\frac{\phi(\alpha_2)-\phi(\alpha_1)}{\Phi(\alpha_2)-\Phi(\alpha_1)}\right] \\ E\left[Y^2|a_1 \le Y \le a_2\right] &= \sigma^2+\mu^2-\sigma^2\left[\frac{\alpha_2\phi(\alpha_2)-\alpha_1\phi(\alpha_1)}{\Phi(\alpha_2)-\Phi(\alpha_1)}\right]-2\mu\sigma\left[\frac{\phi(\alpha_2)-\phi(\alpha_1)}{\Phi(\alpha_2)-\Phi(\alpha_1)}\right] \end{aligned}$$

where

$$\alpha_1 = \frac{a_1-\mu}{\sigma},\quad \alpha_2 = \frac{a_2-\mu}{\sigma}.$$

In what follows we will be interested in computing integrals of the type

$$\begin{aligned} \int_{a_1}^{a_2} yf(y)\,dy &= E[Y|a_1 \le Y \le a_2]\cdot[\Phi(\alpha_2)-\Phi(\alpha_1)] \\ \int_{a_1}^{a_2} y^2 f(y)\,dy &= E\left[Y^2|a_1 \le Y \le a_2\right]\cdot[\Phi(\alpha_2)-\Phi(\alpha_1)] \end{aligned}$$

Therefore, using the theorem above we get the following results:

$$\begin{aligned} \int_{a_1}^{a_2} yf(y)\,dy &= \mu[\Phi(\alpha_2)-\Phi(\alpha_1)]-\sigma[\phi(\alpha_2)-\phi(\alpha_1)] \\ \int_{a_1}^{a_2} y^2 f(y)\,dy &= \left[\sigma^2+\mu^2\right]\cdot[\Phi(\alpha_2)-\Phi(\alpha_1)]-\sigma^2[\alpha_2\phi(\alpha_2)-\alpha_1\phi(\alpha_1)]-2\mu\sigma[\phi(\alpha_2)-\phi(\alpha_1)] \\ \int_{a_1}^{\infty} yf(y)\,dy &= \mu[1-\Phi(\alpha_1)]+\sigma\phi(\alpha_1) \\ \int_{a_1}^{\infty} y^2 f(y)\,dy &= \left[\sigma^2+\mu^2\right]\cdot[1-\Phi(\alpha_1)]+\left[\sigma^2\alpha_1+2\mu\sigma\right]\phi(\alpha_1) \end{aligned}$$

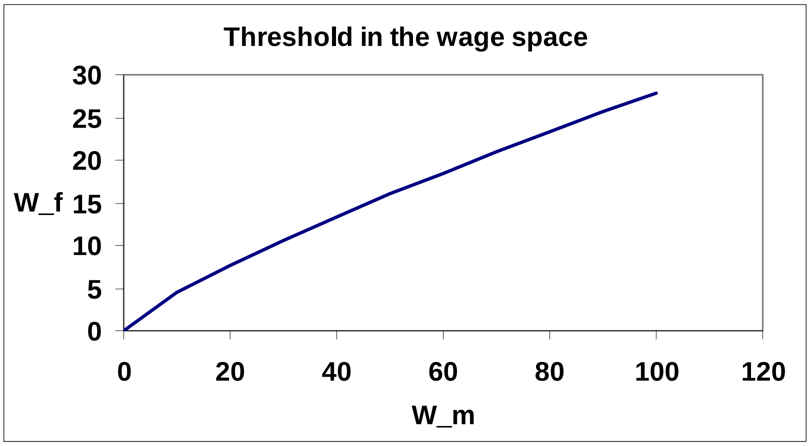
Theorem 3 *Let* $Y \sim N[\mu, \sigma^2]$ *with density* $f(y)$. *Then*

$$\int_{a_1}^{a_2} e^{ty} f(y)\,dy = \exp\left(\mu t+\sigma^2t^2/2\right)\left[\Phi\left(\frac{a_2-\mu}{\sigma}-\sigma t\right)-\Phi\left(\frac{a_1-\mu}{\sigma}-\sigma t\right)\right]$$

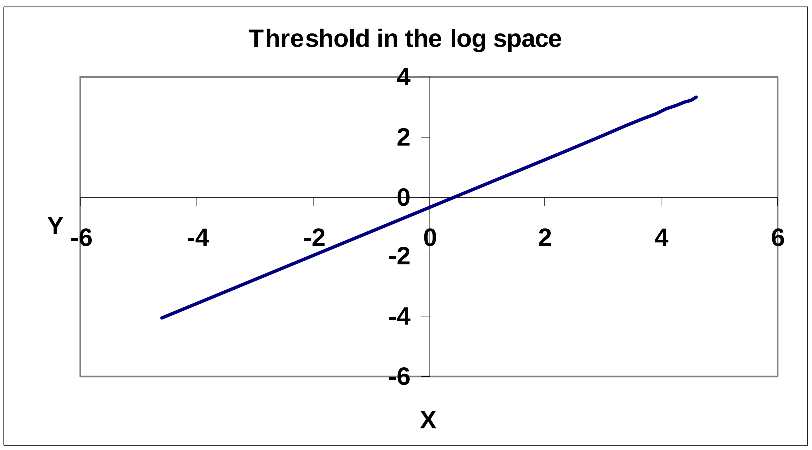
[2] For a proof see Sam Cortum's notes at http://www.econ.umn.edu/~kortum/courses/fall02/lecture4k.pdf

2.3 Computing the Integrals

The model implies a decision rule in a form of a threshold in the wage space, as depicted in the next graph.



In the log space $(X \times Y)$, the above threshold becomes:



The threshold in the log space is a function $l(x)$ such that $V_{2E}\left(e^x, e^{l(x)}\right) = V_{1M}(e^x)$.

Given the parameters of the model, we would like to efficiently compute the following moments:

Moments:
1. $P(1M)$
2. $E[X \vert 1M]$
3. $E[X \vert 2E]$
4. $Var[X \vert 1M]$
5. $Var[X \vert 2E]$
6. $E[Y \vert 2E]$
7. $Var[Y \vert 2E]$
8. $Cov[X, Y \vert 2E]$
9. $P(1M \vert X \leq a)$

1. $P(1M)$

$$
\begin{aligned}
P(1M) &= \int_{-\infty}^{\infty}\int_{-\infty}^{l(x)} f(x,y)\,dydx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{l(x)} f(y|x)\,f_X(x)\,dydx \\
&= \int_{-\infty}^{\infty} f_X(x)\int_{-\infty}^{l(x)} f(y|x)\,dydx \\
&= \int_{-\infty}^{\infty} F(l(x)|x)\,f_X(x)\,dx
\end{aligned}
$$

We can perform the standard transformation

$$
z = \frac{x-\mu_X}{\sigma_X},\ \text{so that } x = z\sigma_X + \mu_X
$$

and let

$$
\alpha(x) = \frac{l(x) - \mu_{Y|X=x}}{\sigma_{Y|X=x}}
$$

Thus,

$$
\begin{aligned}
P(1M) &= \int_{-\infty}^{\infty} \Phi(\alpha(x))\,f_X(x)\,dx \\
&= \int_{-\infty}^{\infty} \Phi(\alpha(x))\,\phi(z)\,dz \\
\text{where } x &= z\sigma_X + \mu_X
\end{aligned}
$$

Observe that $F(l(x)|x) = \Phi(\alpha(x))$.

2. $E[X|1M]$

$$
\begin{aligned}
E[X|1M] &= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{l(x)} xf(x,y)\,dydx}{P(1M)} \\
&= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{l(x)} xf(y|x)\,f_X(x)\,dydx}{P(1M)} \\
&= \frac{\int_{-\infty}^{\infty} xf_X(x)\int_{-\infty}^{l(x)} f(y|x)\,dydx}{P(1M)} \\
&= \frac{\int_{-\infty}^{\infty} xf_X(x)\,\Phi(\alpha(x))\,dx}{P(1M)} \\
&= \frac{\int_{-\infty}^{\infty} x\Phi(\alpha(x))\,\phi(z)\,dz}{P(1M)} \\
\text{where } x &= z\sigma_X + \mu_X
\end{aligned}
$$

3. $E[X|2E]$

$$\begin{aligned} E[X] &= P(1M)\,E[X|1M] + P(2E)\,E[X|2E] \\ \mu_X &= P(1M)\,E[X|1M] + P(2E)\,E[X|2E] \\ E[X|2E] &= \frac{\mu_X - P(1M)\,E[X|1M]}{P(2E)} \end{aligned}$$

4. $Var[X|1M]$

$$\begin{aligned} E\left[X^2|1M\right] &= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{l(x)} x^2 f(x,y)\,dydx}{P(1M)} \\ &= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{l(x)} x^2 f(y|x)\,f_X(x)\,dydx}{P(1M)} \\ &= \frac{\int_{-\infty}^{\infty} x^2 f_X(x)\int_{-\infty}^{l(x)} f(y|x)\,dydx}{P(1M)} \\ &= \frac{\int_{-\infty}^{\infty} x^2 \Phi(\alpha(x))\,f_X(x)\,dx}{P(1M)} \\ &= \frac{\int_{-\infty}^{\infty} x^2 \Phi(\alpha(x))\,\phi(z)\,dz}{P(1M)} \\ \text{where } x &= z\sigma_X + \mu_X \end{aligned}$$

Then we get

$$Var[X|1M] = E\left[X^2|1M\right] - E^2[X|1M]$$

5. $Var[X|2E]$
First observe that

$$\begin{aligned} E\left[X^2\right] &= Var[X] + E^2[X] \\ &= \sigma_X^2 + \mu_X^2 \end{aligned}$$

$$\begin{aligned} E\left[X^2|2E\right] &= \frac{E[X^2] - P(1M)\,E[X^2|1M]}{P(2E)} \\ &= \frac{\sigma_X^2 + \mu_X^2 - P(1M)\,E[X^2|1M]}{P(2E)} \end{aligned}$$

Finally,

$$Var[X|2E] = E\left[X^2|2E\right] - E^2[X|2E]$$

Alternatively,

$$\begin{aligned}
E\left[X^2|1M\right] &= \frac{\int_{-\infty}^{\infty}\int_{l(x)}^{\infty} x^2 f(x,y)\,dydx}{P(2E)} \\
&= \frac{\int_{-\infty}^{\infty}\int_{l(x)}^{\infty} x^2 f(y|x) f_X(x)\,dydx}{P(2E)} \\
&= \frac{\int_{-\infty}^{\infty} x^2 f_X(x)\int_{l(x)}^{\infty} f(y|x)\,dydx}{P(2E)} \\
&= \frac{\int_{-\infty}^{\infty} x^2 f_X(x)\left[1-\Phi(\alpha(x))\right]dx}{P(2E)} \\
&= \frac{\int_{-\infty}^{\infty} x^2\left[1-\Phi(\alpha(x))\right]\phi(z)\,dz}{P(2E)} \\
\text{where } x &= z\sigma_X + \mu_X
\end{aligned}$$

Then,

$$Var[X|2E] = E\left[X^2|2E\right] - E^2[X|2E]$$

6. $E[Y|2E]$

$$\begin{aligned}
E[Y|2E] &= \frac{\int_{-\infty}^{\infty}\int_{l(x)}^{\infty} yf(x,y)\,dydx}{P(2E)} \\
&= \frac{\int_{-\infty}^{\infty}\int_{l(x)}^{\infty} yf(y|x) f_X(x)\,dydx}{P(2E)} \\
&= \frac{\int_{-\infty}^{\infty} f_X(x)\left[\int_{l(x)}^{\infty} yf(y|x)\,dy\right]dx}{P(2E)}
\end{aligned}$$

Let the integral in the brackets be $I_1(x)$. From theorem 2 we have

$$I_1(x) = \mu_{Y|X=x}\left[1-\Phi(\alpha(x))\right] + \sigma_{Y|X=x}\phi(\alpha(x))$$

where

$$\alpha(x) = \frac{l(x) - \mu_{Y|X=x}}{\sigma_{Y|X=x}}$$

Thus,

$$\begin{aligned}
E[Y|2E] &= \frac{\int_{-\infty}^{\infty} I_1(x) f_X(x)\,dx}{P(2E)} \\
&= \frac{\int_{-\infty}^{\infty} I_1(x)\phi(z)\,dz}{P(2E)} \\
\text{where } x &= z\sigma_X + \mu_X
\end{aligned}$$

7. $Var\left[Y|2E\right]$

$$\begin{aligned}
E\left[Y^2|2E\right] &= \frac{\int_{-\infty}^{\infty}\int_{l(x)}^{\infty} y^2 f\left(x,y\right) dydx}{P\left(2E\right)} \\
&= \frac{\int_{-\infty}^{\infty}\int_{l(x)}^{\infty} y^2 f\left(y|x\right) f_X\left(x\right) dydx}{P\left(2E\right)} \\
&= \frac{\int_{-\infty}^{\infty} f_X\left(x\right)\left[\int_{l(x)}^{\infty} y^2 f\left(y|x\right) dy\right] dx}{P\left(2E\right)}
\end{aligned}$$

Let the integral in the brackets be $I_2\left(x\right)$. Then, from theorem 2 we have

$$I_2\left(x\right) = \left[\sigma^2_{Y|X=x} + \mu^2_{Y|X=x}\right]\cdot\left[1-\Phi\left(\alpha\left(x\right)\right)\right] + \left[\sigma^2_{Y|X=x}\alpha\left(x\right) + 2\mu_{Y|X=x}\sigma_{Y|X=x}\right]\phi\left(\alpha\left(x\right)\right)$$

Thus,

$$\begin{aligned}
E\left[Y^2|2E\right] &= \frac{\int_{-\infty}^{\infty} I_2\left(x\right) f_X\left(x\right) dx}{P\left(2E\right)} \\
&= \frac{\int_{-\infty}^{\infty} I_2\left(x\right) \phi\left(z\right) dx}{P\left(2E\right)} \\
\text{where } x &= z\sigma_X + \mu_X
\end{aligned}$$

And

$$Var\left[Y|2E\right] = E\left[Y^2|2E\right] - E^2\left[Y|2E\right]$$

8. $Cov\left[X,Y|2E\right]$

Since $Cov\left[X,Y|2E\right] = E\left[XY|2E\right] - E\left[X|2E\right]E\left[Y|2E\right]$, it is enough to compute $E\left[XY|2E\right]$.

$$\begin{aligned}
E\left[XY|2E\right] &= \frac{\int_{-\infty}^{\infty}\int_{l(x)}^{\infty} xyf\left(x,y\right) dydx}{P\left(2E\right)} \\
&= \frac{\int_{-\infty}^{\infty}\int_{l(x)}^{\infty} xyf\left(y|x\right) f_X\left(x\right) dydx}{P\left(2E\right)} \\
&= \frac{\int_{-\infty}^{\infty} xf_X\left(x\right)\left[\int_{l(x)}^{\infty} yf\left(y|x\right) dy\right] dx}{P\left(2E\right)} \\
&= \frac{\int_{-\infty}^{\infty} xI_1\left(x\right) f_X\left(x\right) dx}{P\left(2E\right)} \\
&= \frac{\int_{-\infty}^{\infty} xI_1\left(x\right) \phi\left(z\right) dz}{P\left(2E\right)} \\
\text{where } x &= z\sigma_X + \mu_X
\end{aligned}$$

9. $P\left(1M|X \leq a\right)$

$$\begin{aligned}
P\left(1M|X\leq a\right) &= \frac{\int_{-\infty}^{a}\int_{-\infty}^{l(x)} f\left(x,y\right)dydx}{P\left(X\leq a\right)}\\
&= \frac{\int_{-\infty}^{a}\int_{-\infty}^{l(x)} f\left(y|x\right)f_X\left(x\right)dydx}{F_X\left(a\right)}\\
&= \frac{\int_{-\infty}^{a} f_X\left(x\right)\int_{-\infty}^{l(x)} f\left(y|x\right)dydx}{F_X\left(a\right)}\\
&= \frac{\int_{-\infty}^{a} F\left(l\left(x\right)|x\right)f_X\left(x\right)dx}{F_X\left(a\right)}
\end{aligned}$$

In particular, we can choose $a = \mu_X$.

10. $P\left(1M|a\leq X\leq b\right)$

For the cross-sectional experiments we need to compute integrals of the form $P\left(1M|a\leq X\leq b\right)$ where a can be $-\infty$ and b can be ∞.

$$\begin{aligned}
P\left(1M|a\leq X\leq b\right) &= \frac{P\left(1M,\ a\leq X\leq b\right)}{P\left(a\leq X\leq b\right)}\\
&= \frac{P\left(Y\leq l\left(x\right),\ a\leq X\leq b\right)}{P\left(a\leq X\leq b\right)}\\
&= \frac{\int_{a}^{b}\int_{-\infty}^{l(x)} f\left(x,y\right)dydx}{F_X\left(b\right)-F_X\left(a\right)}\\
&= \frac{\int_{a}^{b}\int_{-\infty}^{l(x)} f\left(y|x\right)f_X\left(x\right)dydx}{F_X\left(b\right)-F_X\left(a\right)}\\
&= \frac{\int_{a}^{b} f_X\left(x\right)\int_{-\infty}^{l(x)} f\left(y|x\right)dydx}{F_X\left(b\right)-F_X\left(a\right)}\\
&= \frac{\int_{a}^{b} F\left(l\left(x\right)|x\right)f_X\left(x\right)dx}{F_X\left(b\right)-F_X\left(a\right)}
\end{aligned}$$

Notice that all the integrals except for the last two are from $-\infty$ to ∞ with respect to the normal distribution. Therefore, we use the Gauss-Hermite quadrature nodes and weights for those integrals. We prefer to work with the standard normal distribution because then we need to compute the nodes and weights only once. If we had chosen to work with the non-standard normal, then while calibrating, in each iteration we would have to recompute the nodes and weights.

The last two integrals are different in that it is not a full integral. For this integral we use Gauss-Legendre nodes and weights, and we replace ∞ by $8\cdot\sigma_X$ (that is 8 standard deviations from the mean). With this integral we have no choice but to update the nodes and weights in each iteration. Suppose that we wanted to make the change of variable $z = \left(x-\mu_X\right)/\sigma_X$. Then the above integral becomes

$$\int_{-\infty}^{a} F\left(l\left(x\right)|x\right)f_X\left(x\right)dx = \int_{-\infty}^{(a-\mu_X)/\sigma_X} F\left(l\left(z\sigma_X+\mu_X\right)|z\sigma_X+\mu_X\right)\phi\left(z\right)dz$$

But notice that the limit of integration depends on the parameters (which was not the case in full integrals), and since the Gauss-Legendre quadrature nodes and weights depend on the limits of integration, we have to recompute the nodes and weights each time the parameters μ_X and σ_X change.

3 Simulating Random Draws From Normal Distribution

Often we need to simulate a large sample from multivariate normal (or LogNormal) distribution. Sometimes we need to compute an approximation to a moment just once and we don't want to write a code for numerical integration. Or, sometimes it is very difficult to compute the integral numerically, because of high dimensionality of the integrand. Yet another use of random draws is for checking the numerical computation.

Any statistical or mathematical software, has random number generator from the univariate standard normal distribution. So we can obtain a vector of uncorrelated standard normal r.v.'s

$$Z = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$$

where $Z_i \sim N(0,1)$. Now, suppose that we need a random draw from general normal distribution, i.e.,

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Then we let

$$X = \boldsymbol{\mu} + \mathbf{P}Z$$

where $\mathbf{P}$ is the lower Cholesky decomposition of $\boldsymbol{\Sigma}$, so that $\mathbf{PP}' = \boldsymbol{\Sigma}$. To verify that $X \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, recall that X is a linear function of Z, so it has to be normal. The only thing left to do is to compute the mean and variance

$$\begin{aligned} E(X) &= \boldsymbol{\mu} \\ Var(X) &= \mathbf{P}Var(Z)\mathbf{P}' = \mathbf{PIP}' = \boldsymbol{\Sigma} \end{aligned}$$

4 Relating Moments of the Normal and LogNormal

Suppose that $X \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and we define $Y = \exp(X)$, so that Y has LogNormal distribution. In other words,

$$Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} \exp(X_1) \\ \vdots \\ \exp(X_n) \end{bmatrix}$$

We need to find the mean and the covariance matrix of Y. Recall that the moment generating function of a random variable is defined as follows.

$$\psi(\mathbf{t}) = E(\exp(\mathbf{t}X)) = E(\exp(t_1 X_1 + ... + t_n X_n))$$

Let denote the mean vector and the covariance matrix of Y by $\mathbf{m}$ and $\mathbf{S}$ respectively. To find the mean vector observe that

$$E(Y_i) = E(\exp(X_i))$$

Thus, we simply let $t_i = 1$ and $t_j = 0\ \forall j \neq i$ and evaluate the m.g.f. at this $\mathbf{t}$.

To find the second moments observe that

$$E\left(Y_i^2\right) = E\left(e^{X_i} e^{X_i}\right) = E(\exp(2X_i))$$

Thus we let $t_i = 2$ and $t_j = 0\ \forall j \neq i$ and evaluate the m.g.f. at this $\mathbf{t}$. Then the variance of the i^{th} component is obtained by

$$var(Y_i) = E\left(Y_i^2\right) - E^2(Y_i)$$

Finally, the covariance is obtained by

$$\begin{aligned} cov(Y_i, Y_j) &= E(Y_i Y_j) - E(Y_i) E(Y_j) \\ &= E\left(e^{X_i} e^{X_j}\right) - E\left(e^{X_i}\right) E\left(e^{X_j}\right) \\ &= E(\exp(X_i + X_j)) - E(\exp(X_i)) E(\exp(X_j)) \end{aligned}$$

To find the first term on the right we set $t_i = t_j = 1$, and the other coordinates in the vector $\mathbf{t}$ are set to 0.

Recall that the moment generating function of multivariate normal is

$$\psi(\mathbf{t}) = \exp\left(\mathbf{t}'\boldsymbol{\mu} + \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}\right)$$

Therefore, finding moments of the LogNormal distribution is a simple task of evaluating the m.g.f. of the normal distribution at different vectors $\mathbf{t}$, as described above.

4.1 Two variables example

Let $X = [X_1, X_2] \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix},\ \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

Let $Y = [\exp(X_1), \exp(X_2)] \sim LN(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. We want to find the mean vector, $\mathbf{m}$, and the covariance matrix, $\mathbf{S}$, of Y, explicitly written as

$$\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix},\ \mathbf{S} = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}$$

Following the discussion in the previous section we get:
(m_1)

$$\begin{aligned} m_1 &= \psi\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\ &= \exp\left(\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\ &= \exp\left(\mu_1 + \frac{1}{2}\sigma_{11}\right) \end{aligned}$$

(m_2)

$$\begin{aligned} m_2 &= \psi\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= \exp\left(\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= \exp\left(\mu_2 + \frac{1}{2}\sigma_{22}\right) \end{aligned}$$

(s_1^2)

$$\begin{aligned} E\left(Y_1^2\right) &= \psi\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) \\ &= \exp\left(\begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) \\ &= \exp\left(2\mu_1 + 2\sigma_{11}\right) \\ var\left(Y_1\right) &= \exp\left(2\mu_1 + 2\sigma_{11}\right) - \exp\left(2\mu_1 + \sigma_{11}\right) = \exp\left(2\mu_1 + \sigma_{11}\right)\left(\exp\left(\sigma_{11}\right) - 1\right) \end{aligned}$$

(s_2^2)

$$\begin{aligned} E\left(Y_2^2\right) &= \psi\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) \\ &= \exp\left(\begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) \\ &= \exp\left(2\mu_2 + 2\sigma_{22}\right) \\ var\left(Y_2\right) &= \exp\left(2\mu_2 + 2\sigma_{22}\right) - \exp\left(2\mu_2 + \sigma_{22}\right) = \exp\left(2\mu_2 + \sigma_{22}\right)\left(\exp\left(\sigma_{22}\right) - 1\right) \end{aligned}$$

(s_{12})

$$
\begin{aligned}
E(Y_1 Y_2) &= \psi\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \\
&= \exp\left(\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \\
&= \exp\left(\mu_1 + \mu_2 + \sigma_{12} + \frac{1}{2}\sigma_{11} + \frac{1}{2}\sigma_{22}\right) \\
cov(Y_1, Y_2) &= \exp\left(\mu_1 + \mu_2 + \sigma_{12} + \frac{1}{2}\sigma_{11} + \frac{1}{2}\sigma_{22}\right) - \exp\left(\mu_1 + \frac{1}{2}\sigma_{11}\right)\exp\left(\mu_2 + \frac{1}{2}\sigma_{22}\right) \\
&= \exp\left(\mu_1 + \mu_2 + \sigma_{12} + \frac{1}{2}\sigma_{11} + \frac{1}{2}\sigma_{22}\right) - \exp\left(\mu_1 + \mu_2 + \frac{1}{2}\sigma_{11} + \frac{1}{2}\sigma_{22}\right) \\
&= \exp\left(\mu_1 + \mu_2 + \frac{1}{2}\sigma_{11} + \frac{1}{2}\sigma_{22}\right)\left(\exp(\sigma_{12}) - 1\right)
\end{aligned}
$$

Summary of the parameters:

$$
\begin{aligned}
m_1 &= \exp\left(\mu_1 + \frac{1}{2}\sigma_{11}\right) \\
m_2 &= \exp\left(\mu_2 + \frac{1}{2}\sigma_{22}\right) \\
s_1^2 &= \exp(2\mu_1 + \sigma_{11})\left(\exp(\sigma_{11}) - 1\right) \\
s_2^2 &= \exp(2\mu_2 + \sigma_{22})\left(\exp(\sigma_{22}) - 1\right) \\
s_{12} &= \exp\left(\mu_1 + \mu_2 + \frac{1}{2}\sigma_{11} + \frac{1}{2}\sigma_{22}\right)\left(\exp(\sigma_{12}) - 1\right)
\end{aligned}
$$

4.1.1 Solving for μ_i and σ_{ij}

We now solve analytically for the parameters of the underlying normal distribution given the parameters of the LogNormal.

$$
\begin{aligned}
\log m_1 &= \mu_1 + \frac{1}{2}\sigma_{11} \\
\log m_2 &= \mu_2 + \frac{1}{2}\sigma_{22} \\
\log s_1^2 &= 2\mu_1 + \sigma_{11} + \log(\exp(\sigma_{11}) - 1) \\
\log s_2^2 &= 2\mu_2 + \sigma_{22} + \log(\exp(\sigma_{22}) - 1) \\
\log s_{12} &= \mu_1 + \mu_2 + \frac{1}{2}\sigma_{11} + \frac{1}{2}\sigma_{22} + \log(\exp(\sigma_{12}) - 1)
\end{aligned}
$$

(σ_1^2)

$$
\begin{aligned}
\log s_{11} &= 2\mu_1 + \sigma_{11} + \log(\exp(\sigma_{11}) - 1) \\
\log s_{11} &= 2\log m_1 + \log(\exp(\sigma_{11}) - 1) \\
\log\left(\frac{s_{11}}{m_1^2}\right) &= \log(\exp(\sigma_{11}) - 1) \\
\frac{s_{11}}{m_1^2} &= \exp(\sigma_{11}) - 1 \\
\sigma_{11} &= \log\left(1 + \frac{s_{11}}{m_1^2}\right)
\end{aligned}
$$

(σ_2^2)

$$\sigma_{22} = \log\left(1 + \frac{s_{22}}{m_2^2}\right)$$

(μ_1)

$$
\begin{aligned}
\log m_1 &= \mu_1 + \frac{1}{2}\sigma_{11} \\
\mu_1 &= \log m_1 - \frac{1}{2}\log\left(1 + \frac{s_{11}}{m_1^2}\right) \\
\mu_1 &= \log m_1 - \log\left(\frac{\sqrt{m_1^2 + s_{11}}}{m_1}\right) \\
\mu_1 &= \log\left(\frac{m_1^2}{\sqrt{m_1^2 + s_{11}}}\right)
\end{aligned}
$$

(μ_2)

$$\mu_2 = \log\left(\frac{m_2^2}{\sqrt{m_2^2 + s_{22}}}\right)$$

(σ_{12})

$$
\begin{aligned}
\log m_1 &= \mu_1 + \frac{1}{2}\sigma_{11} \\
\log m_2 &= \mu_2 + \frac{1}{2}\sigma_{22} \\
\log m_1 + \log m_2 - \frac{1}{2}\sigma_{11} - \frac{1}{2}\sigma_{22} &= \mu_1 + \mu_2 \\
\log s_{12} &= \log m_1 + \log m_2 + \log\left(\exp(\sigma_{12}) - 1\right) \\
\log s_{12} &= \log\left(m_1 m_2 \left(\exp(\sigma_{12}) - 1\right)\right) \\
s_{12} &= m_1 m_2 \left(\exp(\sigma_{12}) - 1\right) \\
\sigma_{12} &= \log\left(1 + \frac{s_{12}}{m_1 m_2}\right)
\end{aligned}
$$

Summary

$$\begin{aligned}
\mu_1 &= \log\left(\frac{m_1^2}{\sqrt{m_1^2 + s_{11}}}\right) \\
\mu_2 &= \log\left(\frac{m_2^2}{\sqrt{m_2^2 + s_{22}}}\right) \\
\sigma_1^2 &= \log\left(1 + \frac{s_{11}}{m_1^2}\right) \\
\sigma_2^2 &= \log\left(1 + \frac{s_{22}}{m_2^2}\right) \\
\sigma_{12} &= \log\left(1 + \frac{s_{12}}{m_1 m_2}\right)
\end{aligned}$$

4.2 General formulas

Given the Normal random variable $X \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we want to find the mean and covariance matrix of the LogNormal $Y = \exp(X)$. Let the mean vector and covariance matrix of Y be $(\mathbf{m}, \mathbf{S})$

$$\begin{aligned}
E(Y_i) &= \exp\left(\mu_i + \frac{\sigma_{ii}}{2}\right) \\
Var(Y_i) &= \exp(2\mu_i + \sigma_{ii})(\exp(\sigma_{ii}) - 1) \\
Cov(Y_i, Y_j) &= (\exp(\sigma_{ij}) - 1)\exp\left(\mu_i + \mu_j + \frac{\sigma_{ii} + \sigma_{jj}}{2}\right) \\
Cor(Y_i, Y_j) &= \frac{\exp(\sigma_{ij}) - 1}{\sqrt{(\exp(\sigma_{ii}) - 1)(\exp(\sigma_{jj}) - 1)}}
\end{aligned}$$

Given the mean vector and the covariance matrix of the LogNormal distribution, we obtain the mean and covariance matrix of the Normal random variable as follows

$$\begin{aligned}
\mu_i &= E(X_i) = \ln\left(\frac{m_i^2}{\sqrt{m_i^2 + s_{ii}}}\right) \\
\sigma_{ij} &= Cov(X_i, X_j) = \ln\left(1 + \frac{s_{ij}}{m_i m_j}\right)
\end{aligned}$$

4.3 Measures of inequality

The following corollary is following immediately from the above discussion.

Corollary 1 *The two measures of inequality: (1) the coefficient of variation $CV(w) \equiv SD(w)/E(w)$ and (2) $Var(\log(w))$ are always moving in the same direction when w is LogNormally distributed.*

Proof. Suppose that $X = [X_1, X_2] \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $Y = [\exp(X_1), \exp(X_2)] \sim LN(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The moments of X are (mean vector and covariance matrix):

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \ \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix},$$

and the moments of Y are

$$\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \ \mathbf{S} = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}$$

As we showed above,

$$\sigma_{11} = \log\left(1 + \frac{s_{11}}{m_1^2}\right) = \log\left(1 + CV(w)^2\right)$$

The coefficient of variation is always positive, so σ_{11} is monotone increasing in CV. ■

5 Analytical Results

We assume that the household bargaining problem is solved efficiently, so that it can be written as the following social planning problem:

$$\max\{V_{2E}, V_{1M}\}$$

where

$$\begin{aligned} V_{2E} = \max_{c_m^1, c_m^2, c_f^1, c_f^2, k, l_m^2, l_f^2} & \lambda\left[\alpha\left(\log\left(c_m^1\right) + \log\left(c_m^2\right)\right) + (1-2\alpha)\log\left(1 - \bar{l}_m^1 - l_m^2\right)\right] \\ & + (1-\lambda)\left[\alpha\left(\log\left(c_f^1\right) + \log\left(c_f^2\right)\right) + (1-2\alpha)\log\left(1 - \bar{l}_f^1 - l_f^2\right)\right] \\ \text{s.t.} \quad & c_m^1 + c_f^1 + qk \le w_m + w_f, \\ & c_m^2 + c_f^2 \le F\left(k, l_m^2 + l_f^2\right), \\ & 0 \le l_m^2 \le 1 - \bar{l}_m^1 \\ & 0 \le l_f^2 \le 1 - \bar{l}_f^1 \end{aligned}$$

where

$$\begin{aligned} F(k,l) &= \left[\theta k^\rho + (1-\theta) l^\rho\right]^{1/\rho}, \quad -\infty \le \rho \le 1 \\ F(0,l) &= F(k,0) = 0 \text{ if } \rho \le 0. \end{aligned}$$

and V_{1M} is identical to V_{2E} with $w_f = \bar{l}_f^1 = 0$.

Notice that households will always split the market consumption and the consumption of home good such that a fraction λ is consumed by the male and a fraction $(1-\lambda)$ is consumed by the female. In particular, in the 2E case

$$\begin{aligned} c_m^1 &= \lambda(w_m + w_f - qk) \\ c_f^1 &= (1-\lambda)(w_m + w_f - qk) \end{aligned}$$

And the consumption of the home good is divided in a similar way:

$$\begin{aligned} c_m^2 &= \lambda F\left(k, l_m^2 + l_f^2\right) \\ c_f^2 &= (1-\lambda) F\left(k, l_m^2 + l_f^2\right) \end{aligned}$$

Thus, by substituting the above in the maximization problems, and denoting $I_{2E} = w_m + w_f$, we get

$$V_{2E} = \max_{k, l_m^2, l_f^2} \lambda \left[\alpha \log\left(\lambda\left(I_{2E} - qk\right)\right) + \alpha \log\left(\lambda F\left(k, l_m^2 + l_f^2\right)\right) + (1-2\alpha)\log\left(1 - \bar{l}_m^1 - l_m^2\right)\right]$$
$$+ (1-\lambda)\left[\alpha \log\left((1-\lambda)\left(I_{2E} - qk\right)\right) + \alpha \log\left((1-\lambda) F\left(k, l_m^2 + l_f^2\right)\right) + (1-2\alpha)\log\left(1 - \bar{l}_f^1 - l_f^2\right)\right]$$
$$s.t.$$
$$0 \leq k \leq \left(w_m + w_f\right)/q$$
$$0 \leq l_m^2 \leq 1 - \bar{l}_m^1$$
$$0 \leq l_f^2 \leq 1 - \bar{l}_f^1$$

Canceling constants[3]

$$V_{2E} = \max_{k, l_m^2, l_f^2} \lambda \left[\alpha \log\left(I_{2E} - qk\right) + \alpha \log\left(F\left(k, l_m^2 + l_f^2\right)\right) + (1-2\alpha)\log\left(1 - \bar{l}_m^1 - l_m^2\right)\right]$$
$$+ (1-\lambda)\left[\alpha \log\left(I_{2E} - qk\right) + \alpha \log\left(F\left(k, l_m^2 + l_f^2\right)\right) + (1-2\alpha)\log\left(1 - \bar{l}_f^1 - l_f^2\right)\right]$$
$$s.t.$$
$$0 \leq k \leq \left(w_m + w_f\right)/q$$
$$0 \leq l_m^2 \leq 1 - \bar{l}_m^1$$
$$0 \leq l_f^2 \leq 1 - \bar{l}_f^1$$

Collecting terms and plugging the production function $F(k, l) = \left[\theta k^\rho + (1-\theta) l^\rho\right]^{1/\rho}$, gives

$$V_{2E} = \max_{k, l_m^2, l_f^2} \alpha \log\left(I_{2E} - qk\right) + \frac{\alpha}{\rho}\log\left(\theta k^\rho + (1-\theta)\left(l_m^2 + l_f^2\right)^\rho\right)$$
$$+ (1-2\alpha)\left[\lambda \log\left(1 - \bar{l}_m^1 - l_m^2\right) + (1-\lambda)\log\left(1 - \bar{l}_f^1 - l_f^2\right)\right]$$
$$s.t.$$
$$0 \leq k \leq \frac{I_{2E}}{q}$$
$$0 \leq l_m^2 \leq 1 - \bar{l}_m^1$$
$$0 \leq l_f^2 \leq 1 - \bar{l}_f^1$$

[3] We abuse notation and use the same letters to denote the original utility and the transformed utility.

First order conditions:

$$[k] \quad : \quad -\frac{\alpha q}{I_{2E}-qk}+\frac{\alpha\theta k^{\rho-1}}{\theta k^{\rho}+(1-\theta)\left(l_m^2+l_f^2\right)^{\rho}}=0 \tag{1}$$

$$\left[l_f^2\right] \quad : \quad \frac{\alpha(1-\theta)\left(l_m^2+l_f^2\right)^{\rho-1}}{\theta k^{\rho}+(1-\theta)\left(l_m^2+l_f^2\right)^{\rho}}-\frac{(1-2\alpha)(1-\lambda)}{1-\bar{l}_f^1-l_f^2}\leq 0,\text{ with "=" if } l_f^2>0 \tag{2}$$

$$\left[l_m^2\right] \quad : \quad \frac{\alpha(1-\theta)\left(l_m^2+l_f^2\right)^{\rho-1}}{\theta k^{\rho}+(1-\theta)\left(l_m^2+l_f^2\right)^{\rho}}-\frac{(1-2\alpha)\lambda}{1-\bar{l}_m^1-l_m^2}\leq 0,\text{ with "=" if } l_m^2>0 \tag{3}$$

We analyze the case of $\rho\in(0,1]$, i.e. labor and durables are substitutes in the home production. When $\rho<0$, the inputs are complements, so reduction in the price of home appliances will attract more labor into the home production. With inputs being substitutes, the household will trade k for labor in the home production.

Proposition 1 *The optimal k is always interior.*
Proof. *As $k\to 0$ the marginal cost, in terms of utility, of buying extra unit of k is $\alpha q/I_{2E}$, which is finite when $I_{2E}>0$. The marginal benefit, in terms of utility, as $k\to 0$, is $\frac{\alpha\theta k^{\rho-1}}{\theta k^{\rho}+(1-\theta)\left(l_m^2+l_f^2\right)^{\rho}}\to\infty$. Thus, any household with positive income will buy at least some k. To see this, rewrite the limit as*

$$\lim_{k\to 0}\frac{\alpha\theta k^{\rho-1}}{\theta k^{\rho}+(1-\theta)\left(l_m^2+l_f^2\right)^{\rho}}=\lim_{k\to 0}\frac{\alpha\theta}{\theta k+(1-\theta)\left(l_m^2+l_f^2\right)^{\rho}k^{1-\rho}}=\infty$$

Obviously, k will not be at the other corner, I_{2E}/q, because this means that the household will not consume any market good, which is impossible with log utility. ■

Proposition 2 *If $\lambda\geq 1/2$ and $\bar{l}_m^1\geq\bar{l}_f^1$, then at the optimum we have $l_m^2\leq l_f^2$.*

Proof. The marginal benefit, in utility terms, of extra unit of l_m^2 is the same as that of l_f^2 because male and female labor in the home production are perfect substitutes. If men and women worked the same hours at home, say l^2, the marginal cost is higher for men:

$$\frac{(1-2\alpha)\lambda}{1-\bar{l}_m^1-l^2}\geq\frac{(1-2\alpha)(1-\lambda)}{1-\bar{l}_f^1-l^2}$$

The marginal cost of men's work in the house is greater for two reasons: (i) $\lambda\geq 1-\lambda$, and (ii) $\bar{l}_m^1\geq\bar{l}_f^1$. In words, if the weight on male in the utility is higher, then his leisure is more valuable than his wife's. Also, since men work more in the market, they consumes less leisure than their wives, for any given time in the home production. This also makes male's leisure more valuable (more scarce). ■

Corollary 2 *If $l_m^2>0$ then $l_f^2>0$. In other words, the last proposition shows that if male works positive hours in the home production, so does the female. Also, if $l_f^2=0$ then $l_m^2=0$.*

Proposition 3 *In the model with CES home production function, capital-augmenting technological progress in home production is equivalent to declining prices of home appliances.*

Proof. In this version of the model, $F(k,l) = [\theta (Ak)^{\rho} + (1-\theta) l^{\rho}]^{1/\rho}$. We want to show that an increase in A is equivalent to a proportional decrease in q. Define the new variable $\tilde{k} \equiv qk$ and rewrite the problem of the two-earner household as follows.

$$V_{2E} = \max_{c_m^1, c_m^2, c_f^1, c_f^2, \tilde{k}, l_m^2, l_f^2} \lambda \left[\alpha \log\left(c_m^1\right) + \alpha \log\left(c_m^2\right) + (1-2\alpha)\log\left(1 - \bar{l}_m^1 - l_m^2\right)\right]$$
$$+ (1-\lambda)\left[\alpha \log\left(c_f^1\right) + \alpha \log\left(c_f^2\right) + (1-2\alpha)\log\left(1 - \bar{l}_f^1 - l_f^2\right)\right]$$
$$s.t.$$
$$c_m^1 + c_f^1 + \tilde{k} = w_m + w_f$$
$$c_m^2 + c_f^2 \leq \left[\theta\left(A\frac{\tilde{k}}{q}\right)^{\rho} + (1-\theta)\left(l_m^2 + l_f^2\right)^{\rho}\right]^{1/\rho}$$
$$0 \leq l_m^2 \leq 1 - \bar{l}_m^1$$
$$0 \leq l_f^2 \leq 1 - \bar{l}_f^1$$

It is clear that A and q enter in this problem as a fraction. Hence, increasing A by a factor of λ is equivalent to decreasing q by the same factor. The proof is similar for the one-earner male couples. ■

Hence, the experiment of dropping the relative price of home appliances, q, that we discuss in the later section can be interpreted as a capital-augmenting technological change in home production.

6 Elasticity of Substitution in the Home Production

Consider the following model of home production. This is a reduced form household problem, in which the household spends a fraction ν of his income on inputs k and l to the home production.

$$\max_{k,l} [\theta (Ak)^{\rho} + (1-\theta) l^{\rho}]^{1/\rho}$$
$$s.t.$$
$$qk + wl = \nu w$$

$$\max_{\tilde{k},l} \left[\theta\left(A\frac{\tilde{k}}{q}\right)^{\rho} + (1-\theta) l^{\rho}\right]^{1/\rho}$$
$$s.t.$$
$$\tilde{k} + wl = \nu w$$

$$\max_{k,l} [\theta A k^{\rho} + (1-\theta) l^{\rho}]^{1/\rho}$$
$$s.t.$$
$$qk + wl = \nu w$$

Demand for l:

$$\begin{aligned}
l &= \left(\frac{(1-\theta)}{w}\right)^{\sigma} \frac{\nu w}{(\theta A)^{\sigma} q^{\sigma} + (1-\theta)^{\sigma} w^{1-\sigma}} \\
&= (1-\theta)^{\sigma} \frac{\nu w^{1-\sigma}}{(\theta A)^{\sigma} q^{1-\sigma} + (1-\theta)^{\sigma} w^{1-\sigma}} \\
&= (1-\theta)^{\sigma} \frac{\nu}{(\theta A)^{\sigma} (q/w)^{1-\sigma} + (1-\theta)^{\sigma}} \\
&= (1-\theta)^{\sigma} \frac{\nu}{(\theta A^{\rho})^{\sigma} (q/w)^{1-\sigma} + (1-\theta)^{\sigma}} \\
&= (1-\theta)^{\sigma} \frac{\nu}{\theta (A)^{\sigma-1} (q/w)^{1-\sigma} + (1-\theta)^{\sigma}}
\end{aligned}$$

This problem is the dual of minimizing the cost of producing given output:

$$\min_{k,l} qk + wl^2$$

$$s.t.$$

$$\left[\theta A_1 k^{\rho} + (1-\theta) A_2 l^{\rho}\right]^{1/\rho} = \bar{Y}$$

F.O.C.

$$\begin{aligned}
\frac{\theta A_1 k^{\rho-1}}{(1-\theta) A_2 l^{\rho-1}} &= \frac{q}{w} \\
\frac{\theta A_1 l^{1-\rho}}{(1-\theta) A_2 k^{1-\rho}} &= \frac{q}{w} \\
k &= \left[\frac{\theta A_1}{(1-\theta) A_2} \frac{w}{q}\right]^{\frac{1}{1-\rho}} l
\end{aligned}$$

$$\begin{aligned}
\left[\theta A_1 \left(\frac{\theta A_1}{(1-\theta) A_2} \frac{w}{q}\right)^{\frac{\rho}{1-\rho}} l^{\rho} + (1-\theta) A_2 l^{\rho}\right]^{1/\rho} &= \bar{Y} \\
\left[\theta A_1 \left(\frac{\theta A_1}{(1-\theta) A_2} \frac{w}{q}\right)^{\frac{\rho}{1-\rho}} + (1-\theta) A_2\right]^{1/\rho} l &= \bar{Y} \\
\left[\theta A_1^{\sigma} \left(\frac{\theta}{(1-\theta) A_2} \frac{w}{q}\right)^{\sigma-1} + (1-\theta) A_2\right]^{1/\rho} l &= \bar{Y} \\
\left[\theta A_1^{\sigma} \left(\frac{\theta}{(1-\theta)} \frac{w}{q}\right)^{\sigma-1} A_2^{1-\sigma} + (1-\theta) A_2\right]^{1/\rho} l &= \bar{Y} \\
\left[\theta (A_1/A_2)^{\sigma} \left(\frac{\theta}{(1-\theta)} \frac{w}{q}\right)^{\sigma-1} A_2 + (1-\theta) A_2\right]^{1/\rho} l &= \bar{Y} \\
\left[\theta^{\sigma} (A_1/A_2)^{\sigma} (q/w)^{1-\sigma} (1-\theta)^{1-\sigma} A_2 + (1-\theta) A_2\right]^{1/\rho} l &= \bar{Y} \\
\left[\theta^{\sigma} (A_1/A_2)^{\sigma} (q/w)^{1-\sigma} (1-\theta)^{1-\sigma} A_2 + (1-\theta) A_2\right]^{1/\rho} l &= \bar{Y}
\end{aligned}$$

The solution to this problem is

$$
\begin{aligned}
k &= \left(\frac{\theta A_1}{q}\right)^{\sigma} \frac{\nu w}{(\theta A_1)^{\sigma} q^{1-\sigma} + [(1-\theta) A_2]^{\sigma} w^{1-\sigma}} \\
l &= \left(\frac{(1-\theta) A_2}{w}\right)^{\sigma} \frac{\nu w}{(\theta A_1)^{\sigma} q^{1-\sigma} + [(1-\theta) A_2]^{\sigma} w^{1-\sigma}}
\end{aligned}
$$

where $\sigma = \frac{1}{1-\theta}$ is the constant elasticity of substitution. Rearranging so we can see the effect of technological change on the labor employed in the home production.

$$
\begin{aligned}
l &= \left(\frac{(1-\theta) A_2}{w}\right)^{\sigma} \frac{\nu w}{(\theta A_1)^{\sigma} q^{1-\sigma} + [(1-\theta) A_2]^{\sigma} w^{1-\sigma}} \\
&= (1-\theta)^{\sigma} \frac{\nu w^{1-\sigma} A_2^{\sigma}}{(\theta A_1)^{\sigma} q^{1-\sigma} + [(1-\theta) A_2]^{\sigma} w^{1-\sigma}} \\
&= \frac{(1-\theta)^{\sigma} \nu}{(\theta A_1/A_2)^{\sigma} (q/w)^{1-\sigma} + (1-\theta)^{\sigma}}
\end{aligned}
$$

We see that as long as $\sigma > 0$, we have

$$
\begin{aligned}
A_1 &\uparrow \;\Rightarrow l \downarrow \\
A_2 &\uparrow \;\Rightarrow l \uparrow
\end{aligned}
$$

Thus, regardless of whether the two inputs are substitutes or complements, a capital augmenting technological change will decrease the labor employed in the home production, and labor augmenting technological change will increase the labor employed in the home production.

The impact of changes in prices of inputs on the inputs employed, depends crucially on the elasticity of substitution. If the inputs are substitutes ($\sigma > 1$, or $1 - \sigma < 0$) then $q \downarrow$ or $w \uparrow$ will lead to a decline in the labor employed in the home production. So in this simple model the impact of technological improvement is very different from the impact of changes in prices of inputs.

7 Censored Regression Estimation

Formally, the censored regression is described by

$$
\begin{aligned}
Y_i^* &= \mathbf{x}_i \beta + u_i, \quad u_i \sim N\left(0, \sigma^2\right), && (4) \\
Y_i &= \begin{cases} Y_i^* & \text{if } \; Y_i^* \geq R(X_i, \Omega) \\ . & \text{otherwise} \end{cases}, && (5)
\end{aligned}
$$

where Y_i denotes the observed log of the earnings of married female i, and Y_i^* denotes the log of the potential earnings of married female i, which is postulated to depend on her personal attributes $\mathbf{x}_i$, such as years of education, experience, race (see the appendix). A female's potential earnings are observed (Y_i equals Y_i^*), whenever her potential log-earnings exceed her reservation log-earnings $R(X_i, \Omega)$, where X_i is her husband's income. We use " . " to denote unobserved log-earnings.

Thus we are assuming that

$$\begin{aligned} E\left(Y_i^* | \mathbf{x}_i\right) &= \mathbf{x}_i \beta \\ Var\left(Y_i^* | \mathbf{x}_i\right) &= Var\left(u_i\right) = \sigma^2 \end{aligned}$$

Next, we need to compute the probability of not observing the wife's wage.

$$\begin{aligned} \Pr\left(Y_i = .\right) &= \Pr\left(\mathbf{x}_i \beta + u_i < R\left(X_i, \Omega\right)\right) \\ &= \Pr\left(\frac{u_i}{\sigma} \leq \frac{R\left(X_i, \Omega\right) - \mathbf{x}_i \beta}{\sigma}\right) \\ &= \Phi\left(\frac{R\left(X_i, \Omega\right) - \mathbf{x}_i \beta}{\sigma}\right), \end{aligned}$$

where $\Phi(\cdot)$ is the cumulative probability distribution function of the standard normal. Thus, the contribution to the log-likelihood function made by observations with $Y_i = .$ is

$$\log\left(\Phi\left(\frac{R\left(X_i, \Omega\right) - \mathbf{x}_i \beta}{\sigma}\right)\right).$$

Conditional on $Y_i = Y_i^*$, the density of Y_i is (in other words, the conditional density of Y_i given that it is observed):

$$f\left(Y_i | Y_i = Y_i^*\right) = \frac{f\left(Y_i\right)}{\Pr\left(Y_i > 0\right)}$$

Now we can express $f(Y_i)$ in terms of the probability density of standard normal $\phi(\cdot)$.

$$\begin{aligned} f\left(Y_i\right) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{Y_i - \mathbf{x}_i \beta}{\sigma}\right)^2\right] \\ &= \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{Y_i - \mathbf{x}_i \beta}{\sigma}\right)^2\right] \\ &= \sigma^{-1} \phi\left(\frac{Y_i - \mathbf{x}_i \beta}{\sigma}\right) \end{aligned}$$

Now, since when Y_i is observed we have $Y_i = \mathbf{x}_i \beta + u_i$ (the Mincer regression), and $\mu = \mathbf{x}_i \beta$ (u_i has mean zero), we get

$$f\left(Y_i | Y_i = Y_i^*\right) = \frac{\sigma^{-1} \phi\left(\left(Y_i - \mathbf{x}_i \beta\right) / \sigma\right)}{\Pr\left(Y_i = Y_i^*\right)}$$

Since an observation with $Y_i = Y_i^*$ occur with probability $\Pr(Y_i = Y_i^*)$, it's contribution to the likelihood function is

$$\Pr\left(Y_i = Y_i^*\right) f\left(Y_i | Y_i = Y_i^*\right) = \sigma^{-1} \phi\left(\left(Y_i - \mathbf{x}_i \beta\right) / \sigma\right)$$

Thus, the log-likelihood function is:

$$\begin{aligned} l &= \sum_{Y_i=Y_i^*} \log\left(\frac{1}{\sigma}\phi\left(\frac{Y_i - \mathbf{x}_i\beta}{\sigma}\right)\right) + \sum_{Y_i=.} \log\left(\Phi\left(\frac{Y_i - \mathbf{x}_i\beta}{\sigma}\right)\right) \\ &= \sum_{Y_i=Y_i^*} -\frac{1}{2}\left[\log(2\pi) + \log\sigma^2 + \left(\frac{Y_i - \mathbf{x}_i\beta}{\sigma}\right)^2\right] + \sum_{Y_i=.} \log\left(\Phi\left(\frac{Y_i - \mathbf{x}_i\beta}{\sigma}\right)\right). \end{aligned}$$

7.1 The relationship between σ in the censored regression model and the variance in the micro model

The micro model in the paper assumes that the log wages are $[X, Y] \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix}$$

Thus, in the micro model σ_Y^2 is the unconditional variance log of wives wages. We can also compute the conditional variance of Y given X

Theorem 4 *Let* $(X, Y) \sim N\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix}\right)$. *Then*

$$\begin{aligned} (Y|X = x) &\sim N\left[\alpha + \beta x, \sigma_Y^2\left(1 - \rho^2\right)\right] \\ \alpha &= \mu_Y - \beta\mu_X \\ \beta &= \frac{\sigma_{XY}}{\sigma_X^2}, \ \rho = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} \end{aligned}$$

Thus

$$\sigma_{Y|X=x}^2 = \sigma_Y^2\left(1 - \rho^2\right)$$

If in the censored regression model we include only the the log of husband's wage in the attributes, then we could use the variance of the residuals as an estimate $\sigma_{Y|X=x}^2$.

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