Technical Supplement for "The Role of Mortality in the Transmission of Knowledge".

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November 8, 2010

1 Evolution of TFP

In this section we provide additional theoretical results about the behavior of TFPs in our model. Suppose that mortality rates $\{m_{t,j}\}$ are i.i.d. across time and locations, which implies that $B_t = B\ \forall t$. TFP and optimal innovation for the followers is given by:

$$A_{t+1,j} = \left(1 - m_{t,j}^y\right)^{\phi}\left(1 - m_{t-1,j}^y\right)^{\phi}\left[1 + i_{t,j}^{\eta} + \tau\left(\bar{A}_t - A_{t,j}\right)/A_{t,j}\right]A_{t,j} \quad (1)$$

$$\theta\left(1 - i_{t,j}\right)^{\theta(1-\sigma)-1} = B\frac{\eta i_{t,j}^{\eta-1}}{\left[1 + i_{t,j}^{\eta} + \tau\left(\bar{A}_t - A_{t,j}\right)/A_{t,j}\right]^{\sigma}}. \quad (2)$$

where $\bar{A}_t = \max_j\{A_{t,j}\}$ and

$$B_t = \beta E_{\omega_{t,j}}\left[\left(1 - m^y\left(\omega_{t,j}\right)\right)\left[\left(1 - m^y\left(\omega_{t,j}\right)\right)\left(1 - m_{t-1,j}^y\right)\right]^{\phi(1-\sigma)}\left(1 - m^y\left(\omega_{t,j}\right)\right)^{(\theta-1)(1-\sigma)}\right]$$

Let the optimal innovation be $i\left(x_{t,j}\right)$, where $x_{t,j} = \tau\left(\bar{A}_t - A_{t,j}\right)/A_{t,j}$ is the diffusion term. In other words, the difference in innovation activity across locations is due only to the presence of diffusion.

Proposition 1 *Suppose that some location is the leader at time t. Then the expected future path of that location for all $k = 1, 2, ...,$ is given by*

$$E\left(\bar{A}_{t+k}\middle|\left\{A_{t,j}\right\}_{j=1}^{J}, m_{t-1}^y\right) = E\left[\left(1 - m_t^y\right)^{\phi}\right]\cdot E\left[\left(1 - m_t^y\right)^{2\phi}\right]^{k-1}\left(1 - m_{t-1}^y\right)^{\phi}\left[1 + \bar{\imath}^{\eta}\right]^k\bar{A}_t$$

Proof. The law of motion of TFP for a leader is given by:

$$\bar{A}_{t+1} = \left(1 - m_t^y\right)^{\phi}\left(1 - m_{t-1}^y\right)^{\phi}\left[1 + \bar{\imath}^{\eta}\right]\bar{A}_t$$

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Taking conditional expectations, while keeping in mind that $\left\{m_{t,j}^y\right\}$ is i.i.d. across time and locations, gives:

$$E\left(\bar{A}_{t+1}|\left\{A_{t,j}\right\}_{j=1}^J, m_{t-1}^y\right) = E\left[(1-m_t^y)^\phi\right]\left(1-m_{t-1}^y\right)^\phi\left[1+\bar{\imath}^\eta\right]\bar{A}_t$$

Next do the same for $t+2$:

$$\begin{aligned}\bar{A}_{t+2} &= \left(1-m_{t+1}^y\right)^\phi(1-m_t^y)^\phi\left[1+\bar{\imath}^\eta\right]\bar{A}_{t+1} \\ &= \left(1-m_{t+1}^y\right)^\phi(1-m_t^y)^\phi\left[1+\bar{\imath}^\eta\right](1-m_t^y)^\phi\left(1-m_{t-1}^y\right)^\phi\left[1+\bar{\imath}^\eta\right]\bar{A}_t \\ &= \left(1-m_{t+1}^y\right)^\phi(1-m_t^y)^{2\phi}\left(1-m_{t-1}^y\right)^\phi\left[1+\bar{\imath}^\eta\right]^2\bar{A}_t\end{aligned}$$

$$\begin{aligned}E\left(\bar{A}_{t+2}|\left\{A_{t,j}\right\}_{j=1}^J, m_{t-1}^y\right) &= E\left[\left(1-m_{t+1}^y\right)^\phi\right]\cdot E\left[(1-m_t^y)^{2\phi}\right]\cdot\left(1-m_{t-1}^y\right)^\phi\left[1+\bar{\imath}^\eta\right]^2\bar{A}_t \\ &= E\left[(1-m_t^y)^\phi\right]\cdot E\left[(1-m_t^y)^{2\phi}\right]\cdot\left(1-m_{t-1}^y\right)^\phi\left[1+\bar{\imath}^\eta\right]^2\bar{A}_t\end{aligned}$$

Continuing in this fashion gives:

$$\begin{aligned}E\left(\bar{A}_{t+3}|\left\{A_{t,j}\right\}_{j=1}^J, m_{t-1}^y\right) &= E\left[(1-m_t^y)^\phi\right]\cdot E\left[(1-m_t^y)^{2\phi}\right]^2\cdot\left(1-m_{t-1}^y\right)^\phi\left[1+\bar{\imath}^\eta\right]^3\bar{A}_t \\ E\left(\bar{A}_{t+4}|\left\{A_{t,j}\right\}_{j=1}^J, m_{t-1}^y\right) &= E\left[\left(1-m_{t+3}^y\right)^\phi\right]E\left[(1-m_t^y)^{2\phi}\right]^3\left(1-m_{t-1}^y\right)^\phi\left[1+\bar{\imath}^\eta\right]^4\bar{A}_t \\ &\vdots \\ E\left(\bar{A}_{t+k}|\left\{A_{t,j}\right\}_{j=1}^J, m_{t-1}^y\right) &= E\left[(1-m_t^y)^\phi\right]\cdot E\left[(1-m_t^y)^{2\phi}\right]^{k-1}\left(1-m_{t-1}^y\right)^\phi\left[1+\bar{\imath}^\eta\right]^k\bar{A}_t\end{aligned}$$

■

In the next proposition, we establish the behavior of predicted TFPs for the followers.

Proposition 2 . *If $A_{t,i} > A_{t,j}$ then for all $k = 1, 2, ...$ and for all i and j, we have*

$$(i) : E\left(A_{t+k,i}|\left\{A_{t,j}\right\}_{j=1}^J, m_{t-1}^y\right) > E\left(A_{t+k,j}|\left\{A_{t,j}\right\}_{j=1}^J, m_{t-1}^y\right)$$

$$(ii) : \frac{E\left(A_{t+k+1,i}|\left\{A_{t,j}\right\}_{j=1}^J, m_{t-1}^y\right)}{E\left(A_{t+k+1,j}|\left\{A_{t,j}\right\}_{j=1}^J, m_{t-1}^y\right)} < \frac{E\left(A_{t+k,i}|\left\{A_{t,j}\right\}_{j=1}^J, m_{t-1}^y\right)}{E\left(A_{t+k,j}|\left\{A_{t,j}\right\}_{j=1}^J, m_{t-1}^y\right)}$$

$$(iii) : \lim_{k\to\infty}\frac{E\left(A_{t+k,i}|\left\{A_{t,j}\right\}_{j=1}^J, m_{t-1}^y\right)}{E\left(\bar{A}_{t+k}|\left\{A_{t,j}\right\}_{j=1}^J, m_{t-1}^y\right)} = 1$$

*In all the three statements we establish the effect of initial TFP on k-period ahead prediction, **holding the mortality rates at** $t-1$ **fixed**. Therefore we drop the location index in order to make clear that locations i and j differ only by the initial TFP levels, while $m_{t-1,i}^y = m_{t-1,j}^y = m_{t-1}^y$. The first part of the proposition states that if some location is ahead of another at time t, its k-periods ahead predicted value is also greater. The second part states that the distance between predicted TFPs is shrinking. The third part of the proposition states that in the long run, the predicted TFP's of all locations converge to that of the leader.*

Proof. (i). For this part, it is convenient to write the law of motion of TFP as follows:

$$A_{t+1,i} = \left(1 - m^y_{t,i}\right)^\phi \left(1 - m^y_{t-1,i}\right)^\phi \left[\tau \bar{A}_t + \left(1 - \tau + i^\eta_{t,i}\right) A_{t,i}\right]$$

Notice that $A_{t+1,i}$ is an increasing function of $A_{t,i}$ (recall that innovation is increasing in TFP by lemma 2 in the paper, and therefore $A_{t+2,i}$ is increasing in $A_{t+1,i}$, and so on for all $k = 1, 2, ...$ we have $A_{t+k,i}$ is increasing in $A_{t+k-1,i}$. Therefore $A_{t+k,i}$ is increasing in $A_{t,i}$ (since $A_{t+k,i}$ is obtained via composition of increasing functions). This, together with $A_{t,i} > A_{t,j}$ implies that $A_{t+k,i}$ has first order stochastic dominance (f.o.s.d.) over $A_{t+k,j}$, i.e. $\forall a > 0$

$$\Pr\left(A_{t+k,i} > a \middle| \left\{A_{t,j}\right\}_{j=1}^J, m^y_{t-1}\right) > \Pr\left(A_{t+k,j} > a \middle| \left\{A_{t,j}\right\}_{j=1}^J, m^y_{t-1}\right)$$

Since f.o.s.d. implies greater expectation, we obtain:

$$E\left(A_{t+k,i} \middle| \left\{A_{t,j}\right\}_{j=1}^J, m^y_{t-1}\right) > E\left(A_{t+k,j} \middle| \left\{A_{t,j}\right\}_{j=1}^J, m^y_{t-1}\right)$$

(ii). For this part, it is convenient to write the law of motion of TFP as follows:

$$\begin{aligned} A_{t+1,i} &= \left(1 - m^y_{t,i}\right)^\phi \left(1 - m^y_{t-1,i}\right)^\phi \left[1 + i\left(x_{t,i}\right)^\eta + x_{t,i}\right] A_{t,j} \\ \text{where } x_{t,i} &= \tau\left(\bar{A}_t - A_{t,j}\right)/A_{t,j} \end{aligned}$$

Since diffusion is a decreasing function of TFP, we conclude that $x_{t+k,j}$ has first order stochastic dominance over $x_{t+k,i}$, conditional on $A_{t,i} > A_{t,j}$. By proposition **??**, we also conclude that $i\left(x_{t+k,j}\right)^\eta + x_{t+k,j}$ has first order stochastic dominance over $i\left(x_{t+k,i}\right)^\eta + x_{t+k,i}$, which implies that $\forall k = 1, 2, ...$

$$E\left(i\left(x_{t+k,j}\right)^\eta + x_{t+k,j} \middle| \left\{A_{t,j}\right\}_{j=1}^J, m^y_{t-1}\right) > E\left(i\left(x_{t+k,i}\right)^\eta + x_{t+k,i} \middle| \left\{A_{t,j}\right\}_{j=1}^J, m^y_{t-1}\right)$$

This implies that if location j is lagging behind location i at time t, it is *expected* to grow faster in all subsequent periods. In particular, all the time t followers are *expected* to grow faster than the leader in the future.

(iii). Define a sequence of real numbers:

$$E_{k,j} = \left\{ \frac{E\left(A_{t+k,j} \middle| \left\{A_{t,j}\right\}_{j=1}^J, m^y_{t-1}\right)}{E\left(\bar{A}_{t+k} \middle| \left\{A_{t,j}\right\}_{j=1}^J, m^y_{t-1}\right)} \right\}$$

We need to establish that $\{E_{k,j}\}$ is (i) monotone increasing, and (ii) bounded above by 1. Then, by the *monotone convergence theorem* (for sequences of real numbers) it will follow that

$$\lim_{k\to\infty} E_{k,j} = \sup_k \{E_{k,j}\} = 1$$

The sequence $\{E_{k,j}\}$ is monotone increasing by part (ii), which established that any follower's TFP must grow faster than that of a leader. From part (i) of this proposition, the leader is expected to remain the leader, and therefore $\{E_{k,j}\}$ is bounded above by 1, i.e. $\inf_k \{E_{k,j}\} = 1$

$\forall j$. The monotone convergence theorem therefore implies that $\lim_{k\to\infty} E_{k,j} = \sup_k \{E_{k,j}\} = 1$. ∎

Instead of analyzing the behavior of the *expected* TFP, one can ask what will happen to future values of TFPs, if after time t all locations experience the same *fixed* mortality rates. The next proposition presents analogous (and perhaps more intuitive) results to the ones proved in proposition 2, for this deterministic version of the model.

Proposition 3 *Let $\bar{A}_t(J) = \max\{A_{t,i}\}_{i=1}^{J}$ denote the leader's TFP in a J-locations economy, and $|J$ in conditional probability or expectation to denote conditioning on number of locations. The diffusion term then is $x_{t,j}(J) = \tau\left(\bar{A}_t(J) - A_{t,j}\right)/A_{t,j}$. Suppose that all locations have the same initial TFP ($A_{0,j} = A_0 \ \ \forall j$). Then,*

$$E(A_{t,j}|J+1) > E(A_{t,j}|J) \quad \forall j = 1, ..., J$$

The proposition states that the predicted TFPs in all locations increase in the number of locations. The assumption that all locations have the same initial TFP ($A_{0,j} = A_0 \ \ \forall j$) allows us to investigate the impact of number of locations, holding everything else constant (ceteris paribus). This assumption implies that $\{A_{t,i}\}_{i=1}^{J}$ have identical distribution, but not independent of eachother (due to diffusion).

Proof. We compare the distribution of $A_{t,j}|J$ and $A_{t,j}|J+1$, i.e. the distribution of TFP in location j in a J locations economy and the distribution of TFP of the same location j in a $J+1$ locations economy. The TFP in location $J+1$ affects other locations $j = 1, ..., J$ only if it becomes a leader at any time $s = 1, ..., t$, in which case it raises the values of all other locations. Formally, $\forall j = 1, ..., J$ and for all $a > 0$ we have:

$$\Pr(A_{t,j} > a|J+1) \geq \Pr(A_{t,j} > a|J)$$

The inequality is strict if

$$\Pr\left(\{A_{s,J+1} > a\} - \left\{\bigcup_{i=1}^{J}(A_{s,i} > a)\right\}|J+1\right) > 0 \quad \text{for some } s = 1, ..., t$$

Since all locations are ex-ante identical, all of them have the same chance of becoming the leader at any period. Thus we have established that $A_{t,j}|J+1$ has first order stochastic dominance over $A_{t,j}|J$, and since f.o.s.d. implies greater mean, we have that $E(A_{t,j}|J+1) > E(A_{t,j}|J) \quad \forall j = 1, ..., J$. This also implies that $E(A_{t,J+1}|J+1) > E(A_{t,j}|J) \quad \forall j = 1, ..., J$ because all locations have the same distribution of TFP[1]. ∎

Corollary 1 $E\left(\bar{A}_t(J+1)\right) > E\left(\bar{A}_t(J)\right)$

Proof. The expected leader's TFP in a $J+1$ economy must be greater than that of an otherwise identical J locations economy for two reasons: (a) the maximum over $J+1$ random variables is greater than the maximum over J random variables, even when all of them have identical distribution, and (b) in part (i) we showed that with more locations the distribution

[1] All TFPs have identical distribution, but not independent of eachother, because of diffusion.

of each TFP "shifts" in the f.o.s.d. sense, so the maximum is taken over "higher" random variables. Formally, for all $a > 0$ we have

$$\begin{aligned}
\Pr\left(\max\{A_{t,i}\}_{j=1}^{J} > a|J\right) &< \Pr\left(\max\{A_{t,j}\}_{j=1}^{J+1} > a|J\right) \\
&< \Pr\left(\max\{A_{t,j}\}_{j=1}^{J+1} > a|J+1\right)
\end{aligned}$$

The first inequality follows from the fact that the maximum over $J+1$ identically distributed random variables is greater than maximum over any subset of those random variables. The second inequality follows from part (i), which established that TFPs in a $J+1$ locations economy have f.o.s.d. over TFPs in the J locations economy. Therefore, $\bar{A}_t(J+1)$ has f.o.s.d. over $\bar{A}_t(J)$, which implies $E\left(\bar{A}_t(J+1)\right) > E\left(\bar{A}_t(J)\right)$. ■

The above proposition shows that higher number of locations has positive *level* effect on TFPs of all locations. Since all locations started from the same level of TFP, proposition (3) also implies that the predicted growth rate of TFP between period 0 and t is increasing in the number of locations.

The next proposition shows that with no diffusion, the BGP in a deterministic economy and the predicted path in the stochastic economy coinicide, provided that some restrictions hold.

Proposition 4 *Suppose that there is no diffusion, i.e. $\tau = 0$. We have shown that the BGP in the deterministic economy, and the k-periods ahead predicted path for the stochastic economy are given by:*

$$A_{t+k}^d = (1-m^y)^{2k\phi}\left[1+\bar{\imath}^\eta\right]^k A_t^d \tag{3}$$

$$E\left(A_{t+k}^s|A_t^s, m_{t-1}^y\right) = E\left[(1-m_t^y)^\phi\right]\cdot E\left[(1-m_t^y)^{2\phi}\right]^{k-1}\left(1-m_{t-1}^y\right)^\phi\left[1+\bar{\imath}^\eta\right]^k A_t^s \tag{4}$$

The superscripts d and s refer to "deterministic" and "stochastic". If

$$\begin{aligned}
(i) &: A_t^d = A_t^s \\
(ii) &: \left(1-m_{t-1}^y\right)^\phi = E\left[(1-m_t^y)^{2\phi}\right]/E\left[(1-m_t^y)^\phi\right] \\
(iii) &: (1-m^y)^{2\phi} = E\left[(1-m_t^y)^{2\phi}\right]
\end{aligned}$$

then

$$A_{t+k}^d = E\left(A_{t+k}^s|A_t^s, m_{t-1}^y\right) \quad \forall k = 1, 2, ...$$

In other words, under some ristrictions on initial conditions and the constant mortality rate in the deterministic model, the two paths in equations (3) and (4) coincide.

Proof. Notice that the BGP growth rate of TFP in deterministic model is $(1-m^y)^{2\phi}\left[1+\bar{\imath}^\eta\right]$. The growth rate of the predicted path in the stochastic model for $k = 2, 3, ...$ is

$$\begin{aligned}
\frac{E\left(A_{t+k+1}^s|A_t^s, m_{t-1}^y\right)}{E\left(A_{t+k}^s|A_t^s, m_{t-1}^y\right)} &= \frac{E\left[(1-m_t^y)^\phi\right]\cdot E\left[(1-m_t^y)^{2\phi}\right]^k\left(1-m_{t-1}^y\right)^\phi\left[1+\bar{\imath}^\eta\right]^{k+1}A_t^s}{E\left[(1-m_t^y)^\phi\right]\cdot E\left[(1-m_t^y)^{2\phi}\right]^{k-1}\left(1-m_{t-1}^y\right)^\phi\left[1+\bar{\imath}^\eta\right]^k A_t^s} \\
&= E\left[(1-m_t^y)^{2\phi}\right]\left[1+\bar{\imath}^\eta\right]
\end{aligned}$$

However, for $k = 1$ the growth rate of the predicted path is

$$\frac{E\left(A_{t+1}^s|A_t^s, m_{t-1}^y\right)}{A_t^s} = E\left[(1-m_t^y)^\phi\right]\left(1-m_{t-1}^y\right)^\phi[1+\bar{\imath}^\eta]$$

Thus, for the predicted path to exhibit constant growth rate from the very beginning, the initial condition m_{t-1}^y must satisfy

$$\begin{aligned} E\left[(1-m_t^y)^\phi\right]\left(1-m_{t-1}^y\right)^\phi[1+\bar{\imath}^\eta] &= E\left[(1-m_t^y)^{2\phi}\right][1+\bar{\imath}^\eta] \\ \left(1-m_{t-1}^y\right)^\phi &= \frac{E\left[(1-m_t^y)^{2\phi}\right]}{E\left[(1-m_t^y)^\phi\right]} \end{aligned}$$

In other words, for the predicted path to behave like BGP, the initial condition cannot be arbitrary. Substituting the restriction on the initial condition into equation (4) gives

$$E\left(A_{t+k}^s|A_t^s, m_{t-1}^y\right) = E\left[(1-m_t^y)^{2\phi}\right]^k[1+\bar{\imath}^\eta]^k A_t^s$$

Comparison of this with (3) reveals that the two paths are identical, if we set the constant mortality rate in the deterministic model so that is satisfies

$$(1-m^y)^{2\phi} = E\left[(1-m_t^y)^{2\phi}\right]$$

and in addition, $A_t^d = A_t^s$, we have

$$A_{t+k}^d = E\left(A_{t+k}^s|A_t^s, m_{t-1}^y\right), \quad \forall k = 1, 2, ...$$

■

2 Evolution of Consumption

In this section we analyze the impact of mortality on consumption of young adults. Suppose that two identical locations, i and j, experience differential mortality rates $m_{t,j}^y \neq m_{t,i}^y$, but later mortality rates are the same: $m_{t+k,j}^y = m_{t+k,i}^y \;\; \forall k = 1, 2, ...$ What can we say consumption per worker in these locations in the next period and in the long run? Another question is, what is the *ceteris paribus* effect of low mortality in the leader's location on the followers?

Proposition 5 *Suppose that locations i and j are identical at time t, i.e. $A_{t,i} = A_{t,j}$, $c_{t,j}^y = c_{t,i}^y$, $\lambda_{t,j} = \lambda_{t,i}$, and $m_{t-1,j}^y = m_{t-1,i}^y$. In addition, suppose that $m_{t,j}^y > m_{t,i}^y$ and $m_{t+k,j}^y = m_{t+k,i}^y$ $\forall k = 1, 2, ...$ Then:*

$$\begin{aligned} (i) &: m_{t,j} < \frac{(1-\theta)\zeta_t - \phi}{(1-\theta)\zeta_t - \phi\zeta_t} \Rightarrow \frac{c_{t+1,j}^y}{c_{t+1,i}^y} > 1 \\ (ii) &: \lim_{k\to\infty}\frac{c_{t+k,j}^y}{c_{t+k,i}^y} > 1 \end{aligned}$$

Proof. (i) Consumption at $t+1$, and the laws of motion of land and TFP are:

$$\begin{aligned} c^y_{t+1,j} &= A_{t+1,j}\left(1-i\left(x_{t+1,j}\right)\right)^{\theta}\left(\lambda^y_{t+1,j}\right)^{1-\theta} \\ \lambda^y_{t+1,j} &= \frac{0.5\Lambda_j}{N^y_{t+1,j}} = \frac{0.5\Lambda_j}{\left(1-\zeta_t m^y_{t,j}\right) n_{t,j} N^y_{t,j}} \\ A_{t+1,j} &= \left(1-m^y_{t,j}\right)^{\phi}\left(1-m^y_{t-1,j}\right)^{\phi}\left[1+i\left(x_{t,j}\right)^{\eta}+x_{t,j}\right] A_{t,j} \end{aligned}$$

Substituting land and TFP into the production function, and simplifying, gives $c^y_{t+1,j}$ as a function of $c^y_{t,j}$:

$$\begin{aligned} c^y_{t+1,j} &= \left(1-m^y_{t,j}\right)^{\phi}\left(1-m^y_{t-1,j}\right)^{\phi}\left[1+i\left(x_{t,j}\right)^{\eta}+x_{t,j}\right] A_{t,j}\left(1-i\left(x_{t+1,j}\right)\right)^{\theta}\left(\frac{0.5\Lambda_j}{\left(1-\zeta_t m^y_{t,j}\right) n_{t,j} N^y_{t,j}}\right)^{1-\theta} \\ c^y_{t+1,j} &= \frac{\left(1-m^y_{t,j}\right)^{\phi}\left(1-m^y_{t-1,j}\right)^{\phi}\left[1+i\left(x_{t,j}\right)^{\eta}+x_{t,j}\right]}{\left[\left(1-\zeta_t m^y_{t,j}\right) n_{t,j}\right]^{1-\theta}} \frac{\left(1-i\left(x_{t+1,j}\right)\right)^{\theta}}{\left(1-i\left(x_{t,j}\right)\right)^{\theta}} c^y_{t,j} \end{aligned}$$

Comparing consumption in locations j and i at time $t+1$:

$$\begin{aligned} \frac{c^y_{t+1,j}}{c^y_{t+1,i}} &= \frac{\frac{\left(1-m^y_{t,j}\right)^{\phi}\left[1+i\left(x_{t,j}\right)^{\eta}+x_{t,j}\right]}{\left[\left(1-\zeta_t m^y_{t,j}\right) n_{t,j}\right]^{1-\theta}}}{\frac{\left(1-m^y_{t,i}\right)^{\phi}\left[1+i\left(x_{t,i}\right)^{\eta}+x_{t,i}\right]}{\left[\left(1-\zeta_t m^y_{t,i}\right) n_{t,i}\right]^{1-\theta}}} \frac{\frac{\left(1-i\left(x_{t+1,j}\right)\right)^{\theta}}{\left(1-i\left(x_{t,j}\right)\right)^{\theta}}}{\frac{\left(1-i\left(x_{t+1,i}\right)\right)^{\theta}}{\left(1-i\left(x_{t,i}\right)\right)^{\theta}}} \frac{c^y_{t,j}}{c^y_{t,i}} \\ \frac{c^y_{t+1,j}}{c^y_{t+1,i}} &= \frac{\left(1-i\left(x_{t+1,j}\right)\right)^{\theta}}{\left(1-i\left(x_{t+1,i}\right)\right)^{\theta}} \frac{\frac{\left(1-m^y_{t,j}\right)^{\phi}}{\left(1-\zeta_t m^y_{t,j}\right)^{1-\theta}}}{\frac{\left(1-m^y_{t,i}\right)^{\phi}}{\left(1-\zeta_t m^y_{t,i}\right)^{1-\theta}}} \end{aligned}$$

A sufficient condition for $c^y_{t+1,j}/c^y_{t+1,i} > 1$ is that the following function be increasing:

$$f\left(m^y_t\right) = \phi \log\left(1-m^y_{t,j}\right) - (1-\theta)\log\left(1-\zeta_t m^y_{t,j}\right)$$

Taking the derivative

$$f'\left(m^y_t\right) = -\frac{\phi}{1-m^y_{t,j}} + \frac{(1-\theta)\zeta_t}{1-\zeta_t m^y_{t,j}}$$

The derivative is positive when

$$\begin{aligned} -\frac{\phi}{1-m^y_{t,j}} + \frac{(1-\theta)\zeta_t}{1-\zeta_t m^y_{t,j}} &> 0 \\ \frac{(1-\theta)\zeta_t}{1-\zeta_t m^y_{t,j}} &> \frac{\phi}{1-m^y_{t,j}} \\ (1-\theta)\zeta_t - (1-\theta)\zeta_t m^y_{t,j} &> \phi - \phi\zeta_t m^y_{t,j} \\ (1-\theta)\zeta_t - \phi &> (1-\theta)\zeta_t m^y_{t,j} - \phi\zeta_t m^y_{t,j} \\ (1-\theta)\zeta_t - \phi &> (1-\theta-\phi)\zeta_t m^y_{t,j} \\ m_{t,j} &< \frac{(1-\theta)\zeta_t - \phi}{(1-\theta)\zeta_t - \phi\zeta_t} \end{aligned}$$

(ii) By definition, we have

$$c^y_{t+k,j} = A_{t+k,j}\left(1 - i\left(x_{t+k,j}\right)\right)^{\theta}\left(\lambda^y_{t+k,j}\right)^{1-\theta}$$

Comparing locations i and j:

$$\frac{c^y_{t+k,j}}{c^y_{t+k,i}} = \frac{A_{t+k,j}}{A_{t+k,i}} \cdot \frac{\left(1 - i\left(x_{t+k,j}\right)\right)^{\theta}}{\left(1 - i\left(x_{t+k,i}\right)\right)^{\theta}} \cdot \frac{\left(\lambda^y_{t+k,j}\right)^{1-\theta}}{\left(\lambda^y_{t+k,i}\right)^{1-\theta}}$$

We see that the ratio of consumption depends of 3 parts: (a) ratio of TFPs, (b) ratio of labor inputs, and (c) ratio of land inputs. Previous propositions analyzed the evolution of (a) and (b), so now we look at (c), assuming that ζ_t is constant over time.

$$\lambda^y_{t+k,j} = \frac{0.5\Lambda_j}{N^y_{t+k,j}}$$

The evolution of population is

$$\begin{aligned}
N^y_{t+1,j} &= \left(1 - \zeta m^y_{t,j}\right) n_{t,j} N^y_{t,j} \\
N^y_{t+2,j} &= \left(1 - \zeta m^y_{t+1,j}\right) n_{t+1,j}\left(1 - \zeta m^y_{t,j}\right) n_{t,j} N^y_{t,j} \\
&\vdots \\
N^y_{t+k,j} &= \prod_{s=0}^{k-1}\left(1 - \zeta m^y_{t+s,j}\right) \cdot \prod_{s=0}^{k-1} n_{t+s,j} \cdot N^y_{t,j}
\end{aligned}$$

Therefore

$$\lambda^y_{t+k,j} = \left(\frac{1}{\prod_{s=0}^{k-1}\left(1 - \zeta m^y_{t+s,j}\right) \cdot \prod_{s=0}^{k-1} n_{t+s,j}}\right)\lambda^y_{t,j}$$

and

$$\frac{c^y_{t+k,j}}{c^y_{t+k,i}} = \frac{A_{t+k,j}}{A_{t+k,i}} \cdot \frac{\left(1 - i\left(x_{t+k,j}\right)\right)^{\theta}}{\left(1 - i\left(x_{t+k,i}\right)\right)^{\theta}} \cdot \left(\frac{\prod_{s=0}^{k-1}\left(1 - \zeta m^y_{t+s,i}\right) \cdot \prod_{s=0}^{k-1} n_{t+s,i}}{\prod_{s=0}^{k-1}\left(1 - \zeta m^y_{t+s,j}\right) \cdot \prod_{s=0}^{k-1} n_{t+s,j}}\right)^{1-\theta} \cdot \frac{\left(\lambda^y_{t,j}\right)^{1-\theta}}{\left(\lambda^y_{t,i}\right)^{1-\theta}} \qquad (5)$$

Since locations i and j are identical at start of period t, we have $\left(\lambda^y_{t,j}\right)^{1-\theta} / \left(\lambda^y_{t,i}\right)^{1-\theta} = 1$. By proposition 2 in the paper we have $\lim_{k\to\infty} A_{t+k,j}/A_{t+k,i} = 1$, which then implies that $\lim_{k\to\infty} i\left(x_{t+k,j}\right) = \lim_{k\to\infty} i\left(x_{t+k,i}\right) = \bar{\imath}$. If in addition fertility is the same in both locations[2], then the limit of consumption ratio is determined by the limit of the term

$$\left(\frac{\prod_{s=0}^{k-1}\left(1 - \zeta m^y_{t+s,i}\right)}{\prod_{s=0}^{k-1}\left(1 - \zeta m^y_{t+s,j}\right)}\right)^{1-\theta}$$

[2] We are not claiming that in reality the population grows at the same rate in all locations, but rather that logic of *ceteris paribus* requires that we hold everything but mortality rates fixed.

Suppose that location i experienced lower mortality rates than location j for l periods, and after that mortality rates are the same in both locations. Then,

$$\lim_{k\to\infty}\frac{c^y_{t+1,j}}{c^y_{t+1,i}}=\left[\frac{\left(1-\zeta m^y_{t,i}\right)}{\left(1-\zeta m^y_{t,j}\right)}\cdot\frac{\left(1-\zeta m^y_{t+1,i}\right)}{\left(1-\zeta m^y_{t+1,j}\right)}\cdot\ldots\cdot\frac{\left(1-\zeta m^y_{t+l-1,i}\right)}{\left(1-\zeta m^y_{t+l-1,j}\right)}\right]^{1-\theta}$$

Each of the elements in the product is greater than 1, and therefore

$$\lim_{k\to\infty}\frac{c^y_{t+k,j}}{c^y_{t+k,i}}>1$$

■

Discussion. Part (i) of the above proposition established that a one-time higher (lower) mortality in any location, **might** increase (reduce) the consumption per worker in that location, relative to otherwise identical locations. The word "might" is there because the parameters and mortality rates must satisfy the restriction

$$m_{t,j}<\frac{(1-\theta)\,\zeta_t-\phi}{(1-\theta)\,\zeta_t-\phi\zeta_t}$$

Part (ii) on the other hand, establishes that in the long run, a one-time higher (lower) mortality in any location, increases (reduce) the consumption per worker in that location, relative to otherwise identical locations. This statement is true whenever $1-\theta>0$, which follows from definition of labor share.

Proposition 5 does not imply that low temporary mortality rate reduces consumption per capita in **the entire economy**, but only the location with the lower mortality. Suppose that the leader experiences a sequence of low mortality shocks[3], so that the asymptotic balanced growth path in the economy is increases (recall that $\bar{A}_{t+k}=(1-m^y)^{2k\phi}\left[1+\bar{\iota}\eta\right]^k\bar{A}_t$). By proposition 2 in the paper, we have $\lim_{k\to\infty}A_{t+k,i}/\bar{A}_{t+k}=1\ \forall i$, so higher path of TFP for the leader, implies higher path of TFP for the followers, and this leader's impact on the followers increases their **consumption per worker**.

[3] Alternatively, a follower which which expereinces low mortality might become the leader.