Michael Bar ECON605

General equilibrium – pure exchange economy

1. Description of the economy

No production.
Agents (Consumers): {1, 2} with preferences described by $\succsim_i$, $i \in \{1,2\}$.

Goods: {X, Y}

Initial endowment: $\omega = (\omega^1, \omega^2)$, where $\omega^1 = (\omega_x^1, \omega_y^1)$ is the endowment of consumer 1, $\omega^2 = (\omega_x^2, \omega_y^2)$ and $\omega^i = (\omega_x^i, \omega_y^i)$ denotes the amounts of good X and Y that consumer *i* is endowed with, $i \in \{1,2\}$. Notice that $\omega_x^1 + \omega_x^2$ is the total amount of good X in the economy and $\omega_y^1 + \omega_y^2$ is the total amount of good Y in the economy.

Allocation: a collection of consumption bundles that specify how much each agent consumes of each good. That is $(x^1,\ y^1),\ (x^2,\ y^2)$ is an allocation, which specifies that consumer 1 consumes x^1 units of good X and y^1 units of good y, and similarly, $(x^2,\ y^2)$ for consumer 2.

Feasible allocation: the allocation $(x^1,\ y^1),\ (x^2,\ y^2)$ is feasible if

$$x^1 + x^2 = \omega_x^1 + \omega_x^2 \text{ and}$$
$$y^1 + y^2 = \omega_y^1 + \omega_y^2 .$$

That is, the allocation is feasible if the total consumption of X is equal to the total endowment of X and same for Y.

This completes the description of a pure exchange. It is called pure exchange since we ignore the production at this point and concentrate only on how the goods that are *already produced* can be allocated. The problems that we are going to address with this model are

- Among al feasible allocations in the economy, which ones are in some sense “good”? In other words, can we find a criterion that everybody will agree that a “good” allocation should possess? One such criterion is Pareto efficiency, to be discussed later.
- We need to find a mechanism for allocating the goods in the economy. One such mechanism is the *competitive market*. We would like to know weather competitive equilibrium allocations are efficient. The answer to this question is yes, under some weak assumptions.

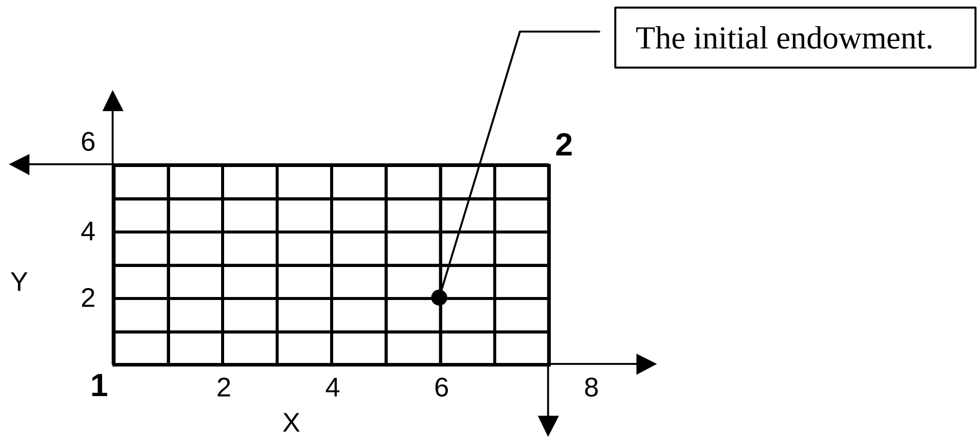
Edgeworth box

A useful graphical tool for describing feasible allocations for economy with 2 agents and 2 goods is the Edgeworth box.

Example: Suppose that the initial endowment is $(\omega_x^1, \omega_y^1) = (6,\ 2)$; $(\omega_x^2, \omega_y^2) = (2,\ 4)$.

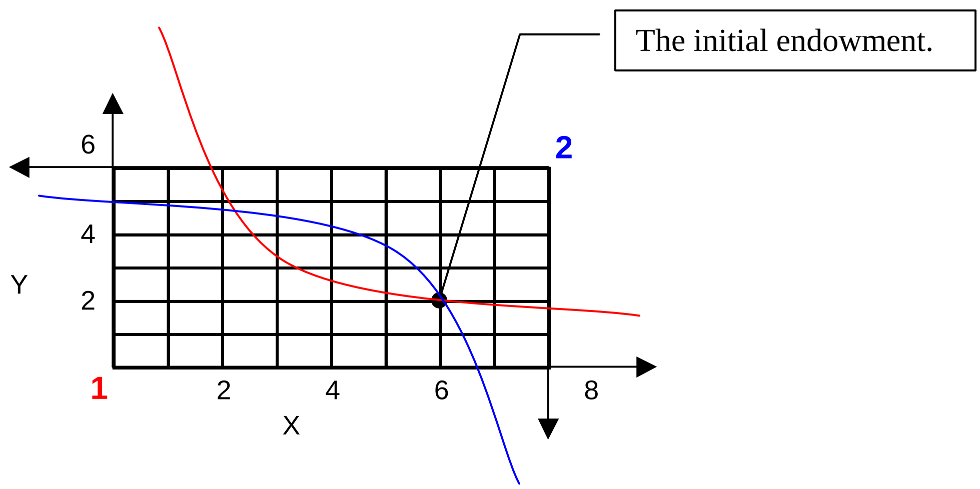
This means that the total amount of good X available in the economy is 8 and the total amount of Y available is 6. The Edgeworth box combines the XY axis of the two consumers such that when we allocate more to consumer 1, there is less available for consumer 2. Figure 1 shows the Edgeworth box for this economy. All the allocations inside the box are feasible. However, the preferences do not depend on the set of feasible allocations. I might prefer Mercedes to BMW, but none of them is feasible for me.

Figure 1



The next figure shows the indifference curves of both consumers. In this example the preferences are convex and increasing in both goods. To distinguish between the two agents, agent 1 is red and agent 2 is blue.

Figure 2



Notice that it is possible to make both consumers better of by moving them to allocation inside the lens area formed by the indifference curves. This suggests that if agents consume their initial endowment, than this allocation is not efficient – there is a waist of resources. Common sense says that if there is a possibility to make both consumers better off, than why not do it? Now we want to be more precise about "wasteful" or inefficient allocation.

2. Pareto Optimal allocations

Definition: Pareto[1] Optimal (or Efficient) allocation.

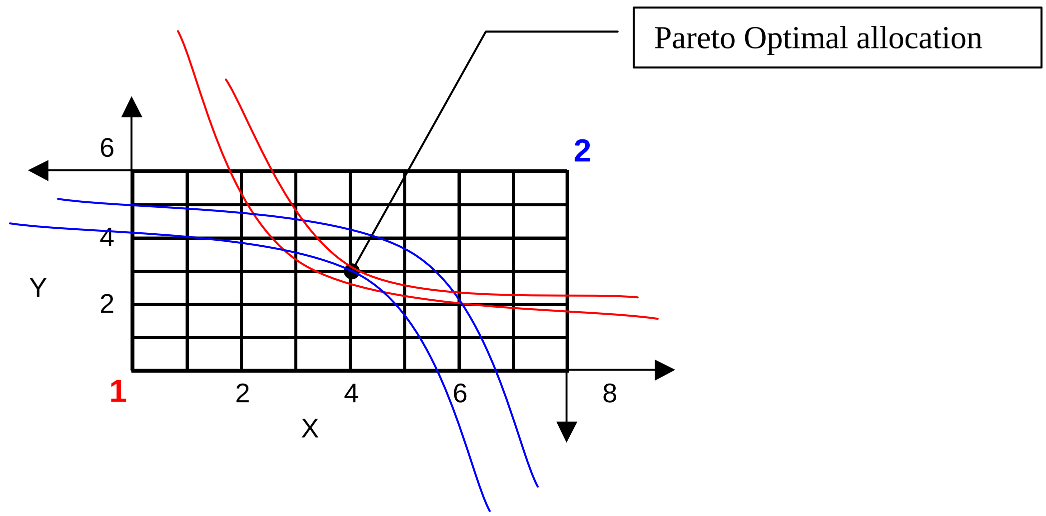
A feasible allocation $x = (x^1, y^1), (x^2, y^2)$ is PO if there is no other feasible allocation $\tilde{x} = (\tilde{x}^1, \tilde{y}^1), (\tilde{x}^2, \tilde{y}^2)$ such that

$$(\tilde{x}^i, \tilde{y}^i) \succsim_i (x^i, y^i) \qquad \text{for all } i \in \{1,2\}$$

$$\text{and} \qquad (\tilde{x}^i, \tilde{y}^i) \succ_i (x^i, y^i) \qquad \text{for some } i \in \{1,2\}.$$

In words, an allocation *x* is PO if it is feasible and there is no other feasible allocation that all consumers weakly prefer and at least one strictly prefers to *x*. In other words, it is impossible to make one agent better off without making the other worse off.
Figure 3 shows one such allocation.

Figure 3



We would like to find the set of all Pareto Optimal allocations, since now we have a criterion that distinguishes between wasteful allocations and efficient allocations. Any allocation that we consider, we might want to check if it is Pareto Optimal or not.

The above figure suggests that the points of tangency between the indifference curves of both agents are PO. This is not always true however. If preferences are not convex then it is not true. Moreover even in the case of strictly convex preferences, the set of Pareto Optimal allocations may contain other points, besides the points of tangencies of indifference curves.

[1] Pareto, Vilfredo (1848.7.15-1923.8.20)

Finding the set of PO allocations.

The set of PO allocations is the solution of the following problem

$$\max_{x^1,y^1} u^1(x^1,y^1)$$
$$s.t.$$
$$u^2(x^2,y^2)=\bar{u}$$
$$x^1+x^2=\omega_x^1+\omega_x^2$$
$$y^1+y^2=\omega_y^1+\omega_y^2$$

The problem says that we want to maximize the utility of agent 1 under the given utility of agent 2. This is equivalent to saying that it is impossible to make agent 1 better off without making agent 2 worse off (the definition of PO). The last two constraints are the feasibility constraints. After all, the definition of Pareto Optimality corresponds only to feasible allocations.

To solve this problem we substitute the feasibility constraints into the first constraint. The problem is

$$\max_{x^1,y^1} u^1(x^1,y^1)$$
$$s.t.$$
$$u^2(\omega_x^1+\omega_x^2-x^1,\omega_y^1+\omega_y^2-y^1)=\bar{u}$$

Denote the total amounts of X and Y in the economy by $\omega_x=\omega_x^1+\omega_x^2, \quad \omega_y=\omega_y^1+\omega_y^2$.

The problem now is

$$\max_{x^1,y^1} u^1(x^1,y^1)$$
$$s.t.$$
$$u^2(\omega_x-x^1,\omega_y-y^1)=\bar{u}$$

Lagrangian

$$L=u^1(x^1,y^1)-\lambda[u^2(\omega_x-x^1,\omega_y-y^1)-\bar{u}]$$

F.O.N.C.

(1) $L_{x^1}=u_x^1(x^1,y^1)+\lambda u_x^2(\omega_x-x^1,\omega_y-y^1)=0$ (the minus sign follows from the chain rule).

(2) $L_{x^1}=u_y^1(x^1,y^1)+\lambda u_y^2(\omega_x-x^1,\omega_y-y^1)=0$.

$u_x^i(x^i,y^i), \quad u_y^i(x^i,y^i)$ denote the partial derivatives of u^i with respect to *x* and *y*.

(1):(2) $\Rightarrow \dfrac{u_x^1(x^1,y^1)}{u_y^1(x^1,y^1)}=\dfrac{u_x^2(\omega_x-x^1,\omega_y-y^1)}{u_y^2(\omega_x-x^1,\omega_y-y^1)}$.

The left hand side is the slope of agent 1 indifference curves (MRS^1) and the right hand side is the slope of agent 2 indifference curves (MRS^2). Thus, if we have "well behaved" preferences, we can find the set of PO allocations using

$$\boxed{MRS^1=MRS^2}$$

Example. Suppose that the initial endowment is $(\omega_x^1, \omega_y^1) = (6,\ 2)$; $(\omega_x^2, \omega_y^2) = (2,\ 4)$.

The preferences are represented by the following utility functions:

$$u^1(x^1, y^1) = x^1 \cdot y^1$$
$$u^2(x^2, y^2) = x^2 \cdot y^2$$

Notice that the upper indexes are not powers. They correspond to the number of the agent, agent 1 and agent 2.

Find the set of all P.O. allocations.

Solution.

$$\boxed{MRS^1 = MRS^2}$$

$$\frac{u_x^1(x^1, y^1)}{u_y^1(x^1, y^1)} = \frac{u_x^2(\omega_x - x^1, \omega_y - y^1)}{u_y^2(\omega_x - x^1, \omega_y - y^1)}$$

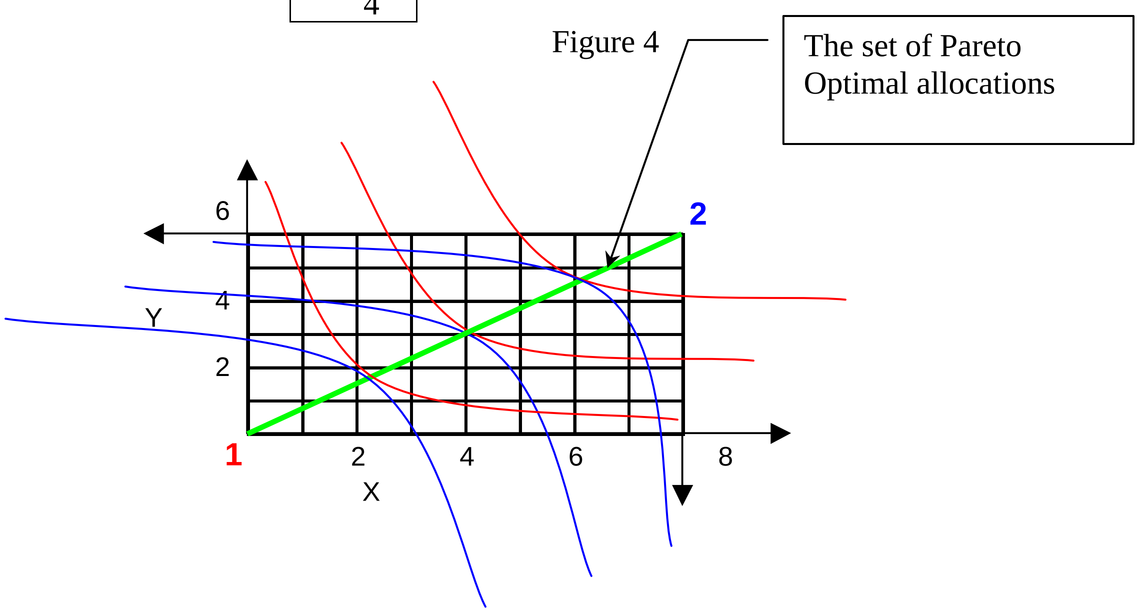
$$\frac{y^1}{x^1} = \frac{y^2}{x^2}$$

$$\frac{y^1}{x^1} = \frac{6 - y^1}{8 - x^1}$$

$$8y^1 - x^1y^1 = 6x^1 - x^1y^1$$

$\boxed{y^1 = \frac{3}{4}x^1}$ (the set of Pareto Optimal allocations).

Figure 4



The set of Pareto Optimal allocations consists of all points of tangency between the agents' indifference curves. Notice that giving all the goods in the economy to one person is efficient in the Pareto sense. However, most people will say that it is "unfair". We never say that all that efficiency is all that matters. But, if we have two alternatives to achieve the same social goal, and one alternative is not efficient while the other is, most people will advise in favor of the efficient alternative. After all, if it is possible to make some people better off, without making the rest worse off, then why not do it?

3. Competitive Equilibrium (Walrasian[2] Equilibrium)

Notice that figure 2 depicts a situation in which the two agents can get better off by trading with each other. Any allocation that is inside the lens shape area makes both consumers better off. For example, the allocation $(5, 3), (3, 3)$ makes both better off (verify that). Recall that the initial endowment is $(\omega_x^1, \omega_y^1) = (6, 2)$; $(\omega_x^2, \omega_y^2) = (2, 4)$. One example of trade between the agents is that agent 1 gives one unit of good X to agent 2 and the latter gives one unit of good Y in return.

Relative prices.
In this economy there is no money, so people trade one good for another (exchange, barter). Nevertheless, we can talk about prices in this economy. We can say for example, that the price of one unit of good X is one unit of good Y. This price is called the relative price of good X. In reality, we observe prices in terms of units of money, but we always can find the relative prices. Suppose that the price of tomatoes is $3 per pound and the price of apples $1.5 per pound. The relative price of tomatoes in terms of apples is 2 (each unit of tomatoes is worth 2 units of apples). Similarly, the relative price of apples in terms of tomatoes is 0.5 (each unit of apples is worth half unit of tomatoes).

The budgets.
Suppose that the prices are in terms of dollars. Then the budgets would look like this

Agent 1 $\quad p_x x^1 + p_y y^1 = p_x \omega_x^1 + p_y \omega_y^1$

Agent 2 $\quad p_x x^2 + p_y y^2 = p_x \omega_x^2 + p_y \omega_y^2$

As usual, the budget has the expenditures on the left hand side and income on the right hand side (Show that this budget is homogeneous of degree zero in prices).
Now, since there is no money in the economy, we need to use relative prices. As a convention, we express the prices of goods in terms of units of good Y. Divide both budgets by p_y to get

Agent 1 $\quad \frac{p_x}{p_y} x^1 + y^1 = \frac{p_x}{p_y} \omega_x^1 + \omega_y^1$

Agent 2 $\quad \frac{p_x}{p_y} x^2 + y^2 = \frac{p_x}{p_y} \omega_x^2 + \omega_y^2$

The price of X in terms of Y is $p = \frac{p_x}{p_y}$ and the price of Y in terms of Y is 1(not surprisingly).

Agent 1 $\quad px^1 + y^1 = p\omega_x^1 + \omega_y^1$

Agent 2 $\quad px^2 + y^2 = p\omega_x^2 + \omega_y^2$

[2] Leon Walras (1834 – 1910).

Definition: Competitive equilibrium.
A competitive equilibrium consists of the prices price ratio *p* and allocation $x = (x^1,\ y^1),\ (x^2,\ y^2)$ such that

1. Given p, the allocation $(x^1,\ y^1),\ (x^2,\ y^2)$ is the best bundle under their budget constraint.
2. The markets are cleared. That is the quantities of X and Y demanded by both consumers is equal to the quantities supplied by both consumers.

The first part says that given the prices, the allocation is in the consumer's demand (since the demand gives us the best choice at any price vector.
The second part of the definition requires that demand = supply in *all markets*, and this is why we call it a *general equilibrium*.

Solving for general equilibrium[3]
Step 1: find the demand[4] of each consumer.
Step 2: equate demand = supply in one[5] of the markets to find the equilibrium prices.
Step 3: plug the prices in each consumer's demand, to find the equilibrium allocation.

Example. Suppose that the initial endowment is $(\omega_x^1, \omega_y^1) = (6,\ 2)$; $(\omega_x^2, \omega_y^2) = (2,\ 4)$.
The preferences are represented by the following utility functions:

$$u^1(x^1, y^1) = x^1 \cdot y^1$$
$$u^2(x^2, y^2) = x^2 \cdot y^2$$

Find the general equilibrium in this economy.
Solution.
Step 1: find the demand[6] of each consumer.
We are familiar with the demand that results form Cobb-Douglass preferences. We know that when the exponents of the utility function are equal, the consumer spends half of his income on each good. Thus the demand is

Consumer 1: $\boxed{x^1(p) = \dfrac{\frac{1}{2}(p \cdot 6 + 2)}{p}}$, $\boxed{y^1(p) = \dfrac{\frac{1}{2}(p \cdot 6 + 2)}{1}}$

Consumer 2: $\boxed{x^2(p) = \dfrac{\frac{1}{2}(p \cdot 2 + 4)}{p}}$, $\boxed{y^2(p) = \dfrac{\frac{1}{2}(p \cdot 2 + 4)}{1}}$

[3] It is important to say that we never proved that the general equilibrium exists. This issue is way beyond the scope of our course, but I will mention that under some week conditions, it exists.
[4] Now you see why I did not want you to forget how to derive the demand.
[5] Wlaras' law implies that if there are n markets, and in (n-1) are in equilibrium, then the last market is also in equilibrium. Later, I prove this result.
[6] Now you see why I did not want you to forget how to derive the demand.

Step 2: equate demand = supply in the market for y.

$$\frac{1}{2}(p\cdot 6+2)+\frac{1}{2}(p\cdot 2+4)=2+4$$

$$\Rightarrow \boxed{p=\frac{3}{4}} \text{ (this is the equilibrium price ratio).}$$

Step 3: plug the prices in each consumer's demand, to find the equilibrium allocation.

Consumer 1: $\boxed{x^1(p)=\frac{\frac{1}{2}(\frac{3}{4}\cdot 6+2)}{\frac{3}{4}}=4\frac{1}{3}}$, $\boxed{y^1(p)=\frac{\frac{1}{2}(\frac{3}{4}\cdot 6+2)}{1}=3\frac{1}{4}}$

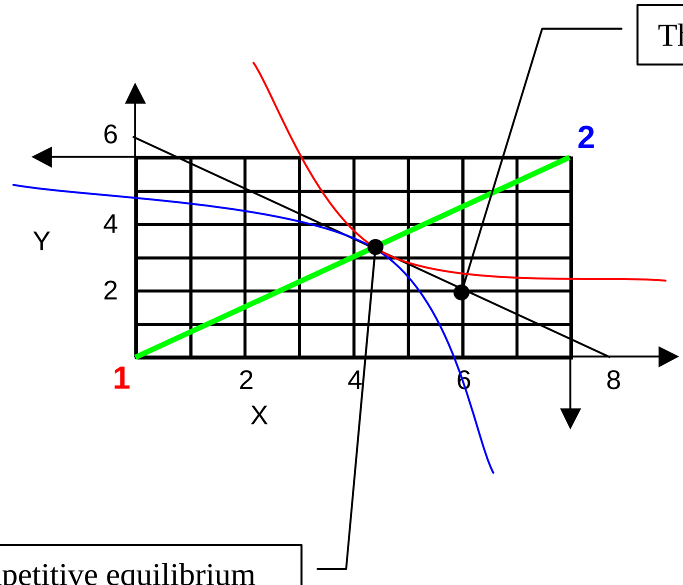
Consumer 2: $\boxed{x^2(p)=\frac{\frac{1}{2}(\frac{3}{4}\cdot 2+4)}{\frac{3}{4}}=3\frac{2}{3}}$, $\boxed{y^2(p)=\frac{\frac{1}{2}(\frac{3}{4}\cdot 2+4)}{1}=2\frac{3}{4}}$

Always check that the equilibrium allocation is feasible. Indeed, $4\frac{1}{3}+3\frac{2}{3}=8,$ and

$3\frac{1}{4}+2\frac{3}{4}=6$.

Now illustrate graphically the equilibrium.

Figure 4



The initial endowment.

Competitive equilibrium allocation

Notice that the competitive equilibrium allocation is Pareto Optimal. This is one of the most important results in welfare economics and called **"The first Fundamental theorem of Welfare Economics"**. It basically means that is agents trade competitively, each acting in his own self interest (maximizes his utility), then the resulting allocation is efficient. This theorem is a formalization of Adam Smith's invisible hand argument and it expresses our confidence in the market economy.