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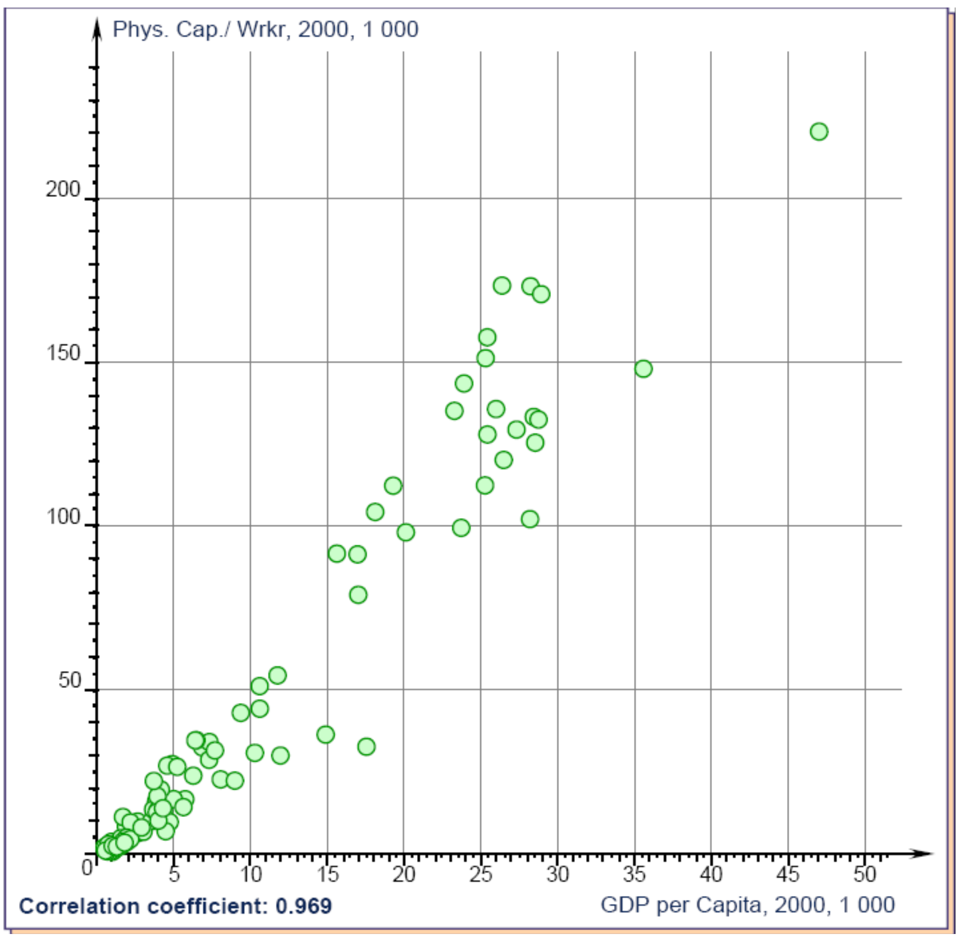
The Role of Physical Capital

As we mentioned in the introduction, the most important macroeconomic observation in the world is the huge differences in output (and income) per capita across countries. The main question of this course is what are the reasons behind these income differences. The basic economic theory suggests that since output is produced using inputs, we should start our search by looking at differences in inputs per worker. Figure 1 shows the relationship between physical capital per worker and GDP per worker for a selection of countries. The figure shows that there is a positive correlation between capital per worker and output per worker.

Figure 1: Physical capital per worker vs. GDP worker in 2000 in selected countries



In the first part of these notes we present a simple theory of aggregate production that can be used in order to decompose the differences across countries in output per capita into three sources: (1) differences in the number of workers per population, (2) differences in productivity, and (3) differences in capital per worker. We will see that differences in capital per worker can indeed account for some of the differences in income per capita across countries. In the second part of these notes we ask why there are differences in capital per worker across countries? We present the Solow model which attempts to answer this question.

1 Accounting for Cross-Country Income Differences

We assume that aggregate output in an economy can be modeled with a Cobb-Douglas production function:

$$Y = AK^{\theta}L^{1-\theta}, \quad 0 < \theta < 1$$

where Y is the total output (GDP), A is the productivity parameter (also called Total Factor Productivity, TFP), K is the stock of physical capital, L is the number of workers, and θ is the capital share[1]. With the assumption of Cobb-Douglas production function, the output per worker is given by

$$y^L = \frac{Y}{L} = \frac{AK^{\theta}L^{1-\theta}}{L} = A\left(\frac{K}{L}\right)^{\theta} = Ak^{\theta}$$

where y^L denotes the output per worker and k denotes the physical capital per worker. Let the population in the country be N and the fraction of workers in the population be

$$\alpha = \frac{L}{N}$$

Thus, output per capita in the country is

$$y^N = \frac{Y}{N} = \frac{\alpha Y}{L} = \alpha y^L$$

This leads to the following accounting formula, that decomposes the ratio of GDP per capita in two countries, i and j, into its attributes:

$$\frac{y_i^N}{y_j^N} = \frac{\alpha_i A_i k_i^{\theta}}{\alpha_j A_j k_j^{\theta}} \tag{1}$$

Equation (1) decomposes the ratio of GDP in country i to that of country j into: (1) α_i/α_j - the ratio of number of workers in population, (2) A_i/A_j - the ratio of productivity, and (3) $k_i^{\theta}/k_j^{\theta}$ - the contribution of capital per worker differences. In this course we will assume that the capital shares in both countries are the same, thus $\theta_i = \theta_j = \theta$. We make this assumption because it is believed that the differences in the measured capital shares across countries is due to measurement error, and in fact all countries have the same capital share.

How to use equation (1)? We typically have data on y^N, α, and k for two countries as in the next table.

	y^N	α	k
Country i	36,000	0.52	160,000
Country j	6,000	0.59	23,000

[1] In the appendix we discuss the properties of this production function.

The ratio of GDP per capita, is 6:1, and we would like to know how much of that is accounted for by differences in capital per worker, assuming that $\theta = 0.35$. Plugging the numbers into equation (1) gives

$$\begin{aligned}\frac{y_i^N}{y_j^N} &= \frac{\alpha_i A_i k_i^\theta}{\alpha_j A_j k_j^\theta} \\ \frac{36,000}{6,000} &= \frac{0.52 A_i (160,000)^{0.35}}{0.59 A_j \, (23,000)^{0.35}}\end{aligned}$$

We see several things from this equation:

1. The difference in the fraction of workers in the population α_i/α_j contributes a factor of 0.88 to the ratio of the GDP per capita of the two countries. This means that if the two countries had the same productivity and the same capital per worker, they would have the same output per worker, but the GDP per capita in country i would be 88% of that in country j. In other words, differences in fraction of workers in population cannot explain why country i is 6 times richer than country j.

2. Country i has 7 times more capital per worker, which contributes a factor of

$$\left(\frac{160,000}{23,000}\right)^{0.35} = 3^{0.35} = 2$$

   to the ratio of GDP per capita in the two countries. This means that if the two countries had the same number of workers per population and the same productivity, then the ratio of GDP per capita would have been 2. Still a large part of the difference between i and j's GDP per capita remains unexplained.

3. How much of the ratio of GDP per capita is accounted for by differences in productivity? Simplifying the accounting equation gives

$$\begin{aligned}\frac{36,000}{6,000} &= \frac{0.52 A_i (160,000)^{0.35}}{0.59 A_j \, (23,000)^{0.35}} \\ &\Rightarrow \frac{A_i}{A_j} = 3.5\end{aligned}$$

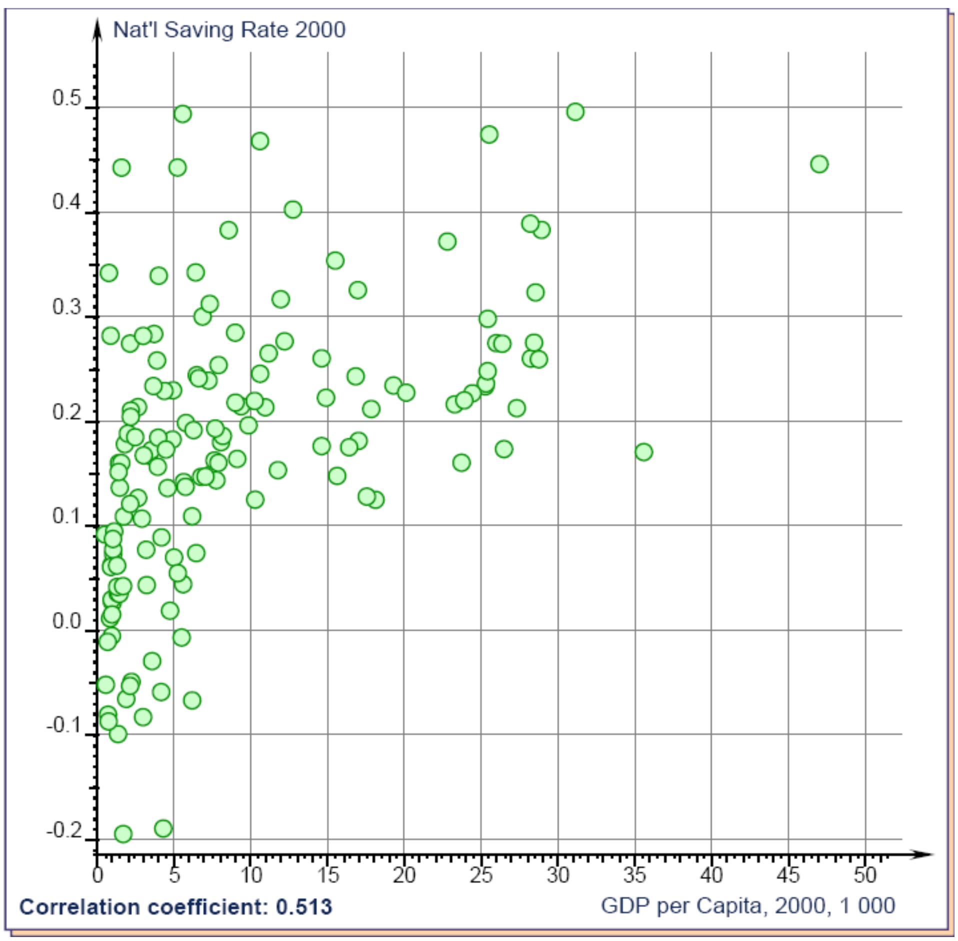
   This means that if the only difference between these countries was the productivity, then the ratio of GDP per capita would have been 3.5. Thus, the largest contribution to the ratio in standard of living, in this example, comes from differences in productivity.

It turns out that this example is typical, and indeed most of the difference between rich and poor countries comes from differences in their productivities. It is not surprising that most of the modern research in the field of economic growth focuses on understanding differences in productivities across countries, and in later chapters we will discuss some of the recent developments. We begin however with a theory that explains differences in capital per worker across countries.

2 The Solow Model

Now that we learned how to decompose the differences in income across countries into differences in capital per worker and other sources, we would like to investigate why there are differences in capital per worker. The Solow model offers a simple mechanism in which people save a constant fraction of their income, and the saving in turn become investment in physical capital. Higher saving rate, all else equal, leads to higher steady state capital per worker. We then can ask a question, how much of the ratio in GDP per capita of two countries can be accounted for by the differences in the national saving rate. Figure (2) shows that there is a positive correlation between GDP per capita and national saving rate.

Figure 2: National Saving Rate vs.GDP per capita in selected countries



2.1 The model description

- Output is produced according to $Y_t = A_t K_t^{\theta} L_t^{1-\theta}, \;\; 0 < \theta < 1.$
- Capital evolves according to $K_{t+1} = K_t (1 - \delta) + I_t$, where δ is the depreciation rate and I_t is aggregate investment.

- People save a fraction s of their income. This fraction is exogenous[2]. Thus, the total saving and total investment in this economy is

$$S_t = I_t = sY_t$$

- The population of workers grows at a constant rate of n, which is exogenous in this model. Thus, $L_{t+1} = (1+n) L_t$.

2.2 Working with the model

Now we derive the predictions of the model. The output per worker is the same as before:

$$y_t^L = \frac{Y_t}{L_t} = \frac{A_t K_t^\theta L_t^{1-\theta}}{L_t} = A_t \left(\frac{K_t}{L_t}\right)^\theta = A_t k_t^\theta$$

The law of motion of capital per worker is

$$\begin{aligned} \frac{K_{t+1}}{L_{t+1}} &= \frac{K_t (1-\delta)}{L_{t+1}} + \frac{I_t}{L_{t+1}} \\ k_{t+1} &= \frac{K_t (1-\delta)}{L_t (1+n)} + \frac{sY_t}{L_t (1+n)} \end{aligned}$$

$$k_{t+1} = \frac{k_t (1-\delta)}{1+n} + \frac{sA_t k_t^\theta}{1+n} \tag{2}$$

Equation (2) describes the law of motion of physical capital per worker. If A_t is fixed at some level A, then the graphical illustration of the law of motion is in figure (3).

With fixed productivity it can be shown that the capital per worker converges to a steady state level, such that

$$k_{t+1} = k_t = k_{ss} \quad \forall t$$

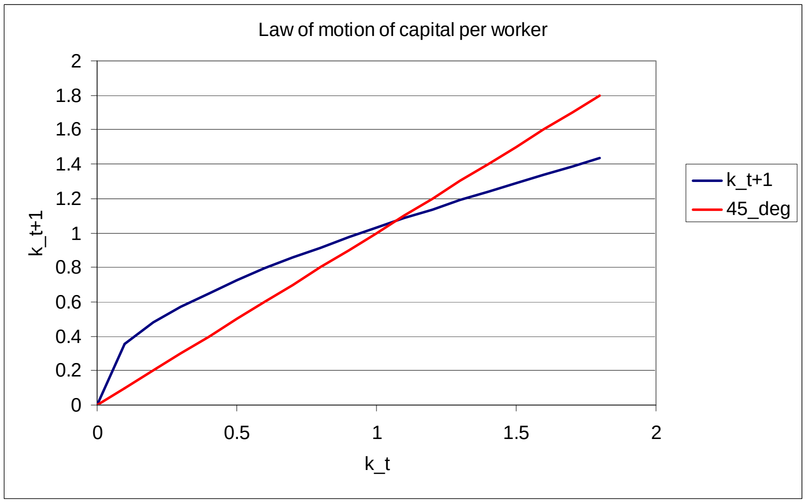
The steady state level of capital per worker can be seen in the graph at the intersection of the law of motion equation with the 45^0 line. It can be shown that starting from any level of capital per worker, k_t converges to the steady state level k_{ss}. Thus, the prediction of the Solow model is that with fixed A, the capital per worker will converge to k_{ss}. This means that in this model, without growth in productivity, there cannot be growth in the standard of living.

2.2.1 Finding the steady state

Using the law of motion and the definition of the steady state

[2] We call a variable **endogenous** if it is determined within the model and **exhogenous** if it is determined outside the model. For example, in the model of a market (supply and demand diagram), the price and quantity traded of the good are endogenous variables, while other variables that determine the location of the supply and demand curve, such as income and prices of other goods, are assumed exogenous.

Figure 3: Law of motion of physical capital per worker.



$$\begin{aligned}
k_{t+1} &= \frac{k_t(1-\delta)}{1+n} + \frac{sAk_t^{\theta}}{1+n} \\
k &= \frac{k(1-\delta)}{1+n} + \frac{sAk^{\theta}}{1+n} \\
k(1+n) &= k(1-\delta) + sAk^{\theta} \\
k(n+\delta) &= sAk^{\theta} \qquad (3) \\
n+\delta &= sAk^{\theta-1}
\end{aligned}$$

The **steady state capital per worker** is

$$k_{ss} = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\theta}} \qquad (4)$$

The intuition for steady state can be seen in equation (3). The term $k(n+\delta)$ represents the "flow out" of capital per worker as a result of depreciation and growth in the number of workers. The term sAk^{θ} represents the "flow in" of the capital per worker due to investment. The steady state requires that those flows cancel each other.

Notice that if the capital per worker is at its steady state (constant) then all other per worker variables are also at their steady state. In particular, the **steady state output per worker** is

$$y_{ss}^{L} = Ak_{ss}^{\theta} = A\left(\frac{sA}{n+\delta}\right)^{\frac{\theta}{1-\theta}} = A^{\frac{1}{1-\theta}}\left(\frac{s}{n+\delta}\right)^{\frac{\theta}{1-\theta}} \qquad (5)$$

The **steady state consumption per worker** is

$$c_{ss} = (1-s)\,Ak_{ss}^{\theta} \qquad (6)$$

The **steady state output per capita** is

$$y_{ss}^N = \alpha A^{\frac{1}{1-\theta}} \left(\frac{s}{n+\delta} \right)^{\frac{\theta}{1-\theta}}$$

Observe that k_{ss} and y_{ss} is increasing in the saving rate and productivity, and decreasing in the population growth rate and depreciation.

2.2.2 Using the Solow model for cross country accounting

We can use the last equation in the previous section to account for cross country differences in GDP per capita, under the assumption that both countries are in the steady state.

$$\frac{y_i^N}{y_j^N} = \frac{\alpha_i A_i^{\frac{1}{1-\theta}} \left(\frac{s_i}{n_i+\delta} \right)^{\frac{\theta}{1-\theta}}}{\alpha_j A_j^{\frac{1}{1-\theta}} \left(\frac{s_j}{n_j+\delta} \right)^{\frac{\theta}{1-\theta}}} \tag{7}$$

If we also assume that the population growth is the same in both countries, then equation (7) reduces to

$$\frac{y_i^N}{y_j^N} = \frac{\alpha_i A_i^{\frac{1}{1-\theta}} (s_i)^{\frac{\theta}{1-\theta}}}{\alpha_j A_j^{\frac{1}{1-\theta}} (s_j)^{\frac{\theta}{1-\theta}}}$$

$$\frac{y_i^N}{y_j^N} = \left(\frac{\alpha_i}{\alpha_j} \right) \left(\frac{A_i}{A_j} \right)^{\frac{1}{1-\theta}} \left(\frac{s_i}{s_j} \right)^{\frac{\theta}{1-\theta}}$$

This decomposes the ratio of GDP per capita into the contribution of: (1) differences in the labor force as a fraction of population, (2) differences in productivity, and (3) differences in the saving rate. Notice that the Solow model accounting puts more weight to the differences in productivity than the simple accounting based only on the Cobb-Douglas production function. In the Solow model accounting, the fraction A_i/A_j is raised to the power of $1/(1-\theta)$, which is about 1.5 when θ is about 1/3. The reason for this is that the TFP parameter A plays two roles: (1) direct role, when higher A means higher output can be produced with the same inputs, and (2) indirect role, when higher A leads to higher steady state level of capital.

2.2.3 Optimal saving rate

Notice that although higher saving rate leads to higher steady state level of capital per worker and output per worker, it does not necessary lead to higher consumption per worker. Observe from equation (6) that on the one hand higher s leads to higher income per worker, but on the other hand higher saving rate means that a smaller fraction of that income is consumed. Now we find the optimal saving rate, i.e. the saving rate that maximizes the steady state consumption per worker. This saving rate is called the **golden rule saving rate**.

$$c_{ss} = (1-s) A k_{ss}^{\theta} = A k_{ss}^{\theta} - (n+\delta) k_{ss}$$

$$\max_{k_{ss}} c_{ss} = Ak_{ss}^{\theta} - (n+\delta) k_{ss}$$

First order condition:

$$\begin{aligned} \theta A k_{GR}^{\theta-1} &= n+\delta \\ k_{GR} &= \left(\frac{\theta A}{n+\theta}\right)^{\frac{1}{1-\theta}} \end{aligned}$$

Now comparing this with the steady state capital

$$k_{ss} = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\theta}}$$

implies that

$$s_{GR} = \theta$$

3 Appendix: Producer's Choice

We represented the consumer's preferences with a utility function. In a similar fashion we represent the production technology with a production function. We assume that there are two inputs, capital (K) and labor (L).

Definition 1 *A* ***production function*** $F(K, L)$ *gives the maximal possible output that can be produced when using* K *units of capital and* L *units of labor.*

Example. A widely used production function in economics is the Cobb-Douglas production function

$$Y = AK^{\theta}L^{1-\theta}, \quad 0 < \theta < 1$$

where Y is the output, A is productivity parameter, K is the capital, L is labor, and θ is called the capital share in output. We will discuss this parameter later. Suppose that $A = 10,\ K = 7,\ L = 20,\ \theta = 0.35$. What is the maximal output that can be produced with this technology and these inputs? Answer:

$$Y = 10 \cdot 7^{0.35} \cdot 20^{1-0.35} \approx 138.5$$

Definition 2 *A production* $F(K, L)$ *exhibits* ***constant returns to scale*** *if*

$$F(\lambda K, \lambda L) = \lambda F(K, L), \quad \forall \lambda > 0$$

This means if a function exhibits constant returns to scale, then when we double all the inputs, the output is also doubled. To see this let $\lambda = 2$ in the above definition. Then

$$F(2K, 2L) = 2F(K, L)$$

Example. The Cobb-Douglas production function exhibits constant returns to scale.

$$A(\lambda K)^{\theta}(\lambda L)^{1-\theta} = \lambda^{\theta}\lambda^{1-\theta}AK^{\theta}L^{1-\theta} = \lambda AK^{\theta}L^{1-\theta}$$

Definition 3 *The marginal product of capital is* $F_K(K,L)$ *and the marginal product of labor is* $F_L(K,L)$*.*

Thus, the marginal product of each input is the partial derivative of the production function with respect to the input. In words, the marginal product of capital is the change in total output that results from a small change in capital input. The marginal product of labor is the change in the total output that results from a small change in the labor input. The marginal product is completely analogous to the marginal utility in the consumer theory.

3.1 Firm's profit maximization problem

We assume that the firm is competitive in both the output market and the input markets. That is, the firm takes the prices of output and inputs as given. Let P be the price of the output, and W and R be the wage and the rental rate of capital respectively. The firm's maximization problem is

$$\begin{aligned} & \max_{Y,K,L} P \cdot Y - RK - WL \\ & s.t. \\ Y &= F(K,L) \end{aligned}$$

Hence the firm chooses the output level and how much inputs to employ, and it maximizes profit subject to the technology constraint. The term $P \cdot Y$ is the revenue and therefore $P \cdot Y - RK - WL$ is the profit (revenue - cost). In applications to Macro, we will often express all the magnitudes in real terms. Thus we will divide the profit by the price level P and denote the real prices of inputs by

$$r = \frac{R}{P}, \quad w = \frac{W}{P}$$

The profit maximization problem then becomes

$$\begin{aligned} & \max_{Y,K,L} Y - rK - wL \\ & s.t. \\ Y &= F(K,L) \end{aligned}$$

The easiest way to solve the profit maximization problem is to substitute the constraint into the objective

$$\max_{K,L} F(K,L) - rK - wL$$

This is an unconstrained optimization problem, and the only choice of the firm is the quantities of inputs K and L. The first order conditions for optimal input mix:

$$\begin{aligned} F_K(K,L) - r &= 0 \\ F_L(K,L) - w &= 0 \end{aligned}$$

or

$$\begin{aligned} F_K(K,L) &= r \\ F_L(K,L) &= w \end{aligned}$$

Thus, a competitive firm pays each input its marginal product.

Example. Write the profit maximization problem for a competitive firm with Cobb-Douglas technology and derive the first order conditions for optimal input mix.

Answer:

$$\text{Profit maximization problem: } \max_{K,L} AK^{\theta}L^{1-\theta} - rK - wL$$

First order conditions for optimal input mix:

$$\begin{aligned} \theta AK^{\theta-1}L^{1-\theta} &= r \\ (1-\theta)\,AK^{\theta}L^{-\theta} &= w \end{aligned}$$

Thus, in competition the rental rate of capital is equal to the marginal product of capital and the wage is equal to the marginal product of labor.

3.2 Factor shares

Suppose that the aggregate output in the economy (real GDP) can be modeled as Cobb-Douglas production function:

$$Y = AK^{\theta}L^{1-\theta}$$

And suppose that each input is paid its marginal product. Then a fraction θ of the total output is paid to capital and a fraction $(1-\theta)$ of the total output is paid to labor. To see this notice that the payment to capital is

$$\begin{aligned} rK &= \theta AK^{\theta-1}L^{1-\theta}\cdot K = \theta AK^{\theta}L^{1-\theta} = \theta Y \\ \text{thus } \frac{rK}{Y} &= \theta \end{aligned}$$

and the payment to labor is

$$\begin{aligned} wL &= (1-\theta)\,AK^{\theta}L^{-\theta}\cdot L = (1-\theta)\,AK^{\theta}L^{1-\theta} = (1-\theta)\,Y \\ \text{thus } \frac{wL}{Y} &= 1-\theta \end{aligned}$$