

Problem set 1 - Solution

The Mathematics of Growth Rate

1. Suppose that you deposit a certain amount of money in a savings account with interest rate of 2% per year.
 - a. How long will it take for your money to double? Show your calculations. You are not allowed to use any approximation formulas, such as the “rule of 70”.

Let y_0 be the initial amount of your deposit. Doubling this amount means that after certain time you have $2y_0$. Thus, we need to solve the following equation for the unknown t :

$$y_0(1.02)^t = 2y_0$$

Notice that the initial amount cancels out, so the equation becomes $(1.02)^t = 2$, and the only unknown is t . This means that the time that takes your saving to double does not depend on how much money you have in the beginning, but only depends on the interest rate (how fast your savings are growing). Taking \ln 's of both sides

$$t \ln(1.02) = \ln(2)$$

$$t = \frac{\ln 2}{\ln 1.02} \approx 35.00279$$

- b. Calculate the approximate doubling using the “rule of 70”.

$$t \cong \frac{70}{2} = 35$$

2. Suppose that U.S. GDP is twice as large as the GDP of China. Suppose that the U.S. GDP grows on average at 2% per year, while the Chinese GDP grows at 7% per year. How many years would it take for China to catch up with the U.S. in terms of GDP?

Let x be the GDP in China, so that $2x$ is the U.S. GDP. If after t years China catches up, we need to solve the following equation for the unknown t :

$$x \cdot (1.07)^t = 2x \cdot (1.02)^t$$

$$t \ln(1.07) = \ln 2 + t \ln(1.02)$$

$$t [\ln(1.07) - \ln(1.02)] = \ln 2$$

$$t = \frac{\ln 2}{[\ln(1.07) - \ln(1.02)]} = 14.48401$$

3. Suppose that the GDP per capita in the U.S. is 5 times that of China. Assuming that the U.S. has been growing on average at 2% per year, how many years ago did the U.S. have the same GDP per capita as China today?

Let x be the GDP per capita of China today, and in U.S. t years ago. For the last t years the U.S. is growing at 2% per year, so today the GDP per capita in U.S. is $5x$. Thus, the equation to be solved for the unknown t is:

$$\begin{aligned}x \cdot (1.02)^t &= 5x \\t \ln(1.02) &= \ln 5 \\t &= \frac{\ln 5}{\ln(1.02)} = 81.27396\end{aligned}$$

This means that the standard of living in China today is the same as in the U.S. approximately 81 years ago.

4. Suppose that the price of the firm's output has increased by 1.5%, and the quantity sold increased by 2.5%. What is the approximate growth rate of the firm's revenue?

$$\hat{R} = 1.5\% + 2.5\% = 4\%$$

5. Suppose that the GDP of some country grows at 3% per year, and its population grows at 2.5% per year. What is the approximate growth rate of GDP per capita in this country?

$$\hat{y} = 3\% - 2.5\% = 0.5\%$$

6. Suppose that some variable is growing at constant rate.
 - a. Prove that the natural logarithm of that variable is a linear function of time.

If a variable is growing at constant rate g , then its value at time t is given by

$$y_t = y_0(1 + g)^t$$

Taking logs of that

$$\ln y_t = \ln y_0 + t \ln(1 + g)$$

Which is a linear function of t .

- b. Find the intercept and slope of the linear function in part a.

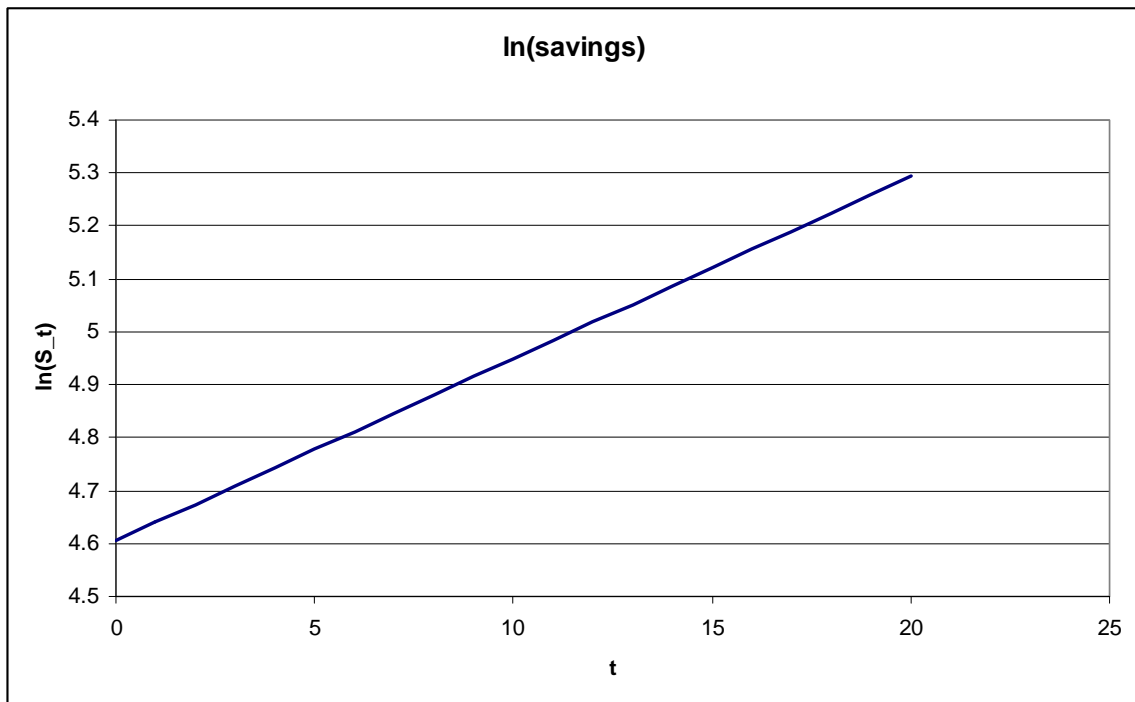
The intercept is $\ln(y_0)$ and the slope is $\ln(1 + g)$.

7. This question illustrates that when a variable grows at constant rate, then the graph of the \ln of the variable is a linear function of time, with slope that is approximately equal to the growth rate of the original variable (when that growth rate is small). Suppose that you put 100\$ in a savings account at annual interest rate of 3.5%. Let S_t be the amount of savings at time t , where $t = 0, 1, \dots, 20$.

- a. Using Excel, plot the graph that shows the amount of savings that you have in each of the years $t = 0, 1, \dots, 20$. That is, plot S_t against t .



- b. Using Excel, plot the graph of $\ln(S_t)$ against t .



- c. What can you say about the shape of the graph in part b? What is the slope of the graph in part b?

The $\ln(S_t)$ is a linear function of time. The slope of the line is $\ln(1.035) \cong 0.0344$, which is approximately the (net) growth rate of the original variable S_t .

8. Based on the following graph, what is the approximate growth rate of the variable y_t ?



- a. 2.3%
b. 10%
 c. 3%
d. 6%

To see why, recall that the slope of the $\ln(y_t)$ is approximately equal to the growth rate of y_t . The slope of the above line is $\Delta Y / \Delta X = (2.6 - 2.3) / 10 = 0.3 / 10 = 0.03 = 3\%$.

Introduction

9. Based on the graphs from the “Introduction” slides that displays the real GDP per capita and the ln of that variable in different countries (ARG = Argentina, CHN = China, KOR = South Korea, TCD = Chad, USA = USA).

a. Are all the above countries experiencing sustained growth in real GDP per capita?

No. Argentina had negative growth in real GDP per capita during the 80’s. Chad experienced negative growth most of the time.

b. What can you say about the growth rate of real GDP per capita in the USA and Argentina prior to 1975? Explain how you reached your conclusion.

About the same. By looking at the ln(real GDP per capita) of both countries, we observe that before the mid seventies, the two trends were approximately parallel and straight lines. This means that the slopes of the trends were the same, and thus the growth rate was about the same.

10. This is a small research project. Visit the CIA World Factbook. The following link shows the ranking of 228 countries/locations according to GDP per capita, PPP adjusted: <https://www.cia.gov/library/publications/the-world-factbook/rankorder/2004rank.html>

Choose 5 countries among the 20 poorest and 5 countries among the 20 richest, based on GDP per capita. For each country report (1) GDP per capita, (2) TFR (Total Fertility Rate, which is the total number of children that a woman is expected to have during her lifetime), (3) life expectancy, and (4) one more variable that you think is interesting. Present your results in two tables, one for the 5 rich countries and one for the 5 poor countries, with the averages for each group of countries. What did you learn from this mini-research?

Here is how I would present the results:

5 Poor Countries:

	GDP/capita	TFR	Life Expectancy	Literacy
Zimbabwe	\$ 200	3.69	45.77	90.7%
Congo	\$ 300	6.20	54.36	67.2%
Afghanistan	\$ 800	5.60	44.40	28.1%
Rwanda	\$ 1,000	5.12	56.77	70.4%
Haiti	\$ 1,300	3.81	60.78	52.9%
Average	\$720	4.884	52.416	61.86%

5 Rich Countries:

	GDP/capita	TFR	Life Expectancy	Literacy
Norway	\$59,300	1.78	79.95	100%
Singapore	\$ 50,300	1.09	81.98	92.5%
U.S.	\$ 46,400	2.05	78.11	99%
Hong Kong	\$ 42,700	1.02	81.86	93.5%
Ireland	\$ 42,200	1.85	78.24	99%
Average	\$48,180	1.558	80.0	97%

I learnt that poor countries/regions tend to have high fertility, low life expectancy and low literacy. I was surprised by the high literacy rate in Zimbabwe – to poorest country in the world today.