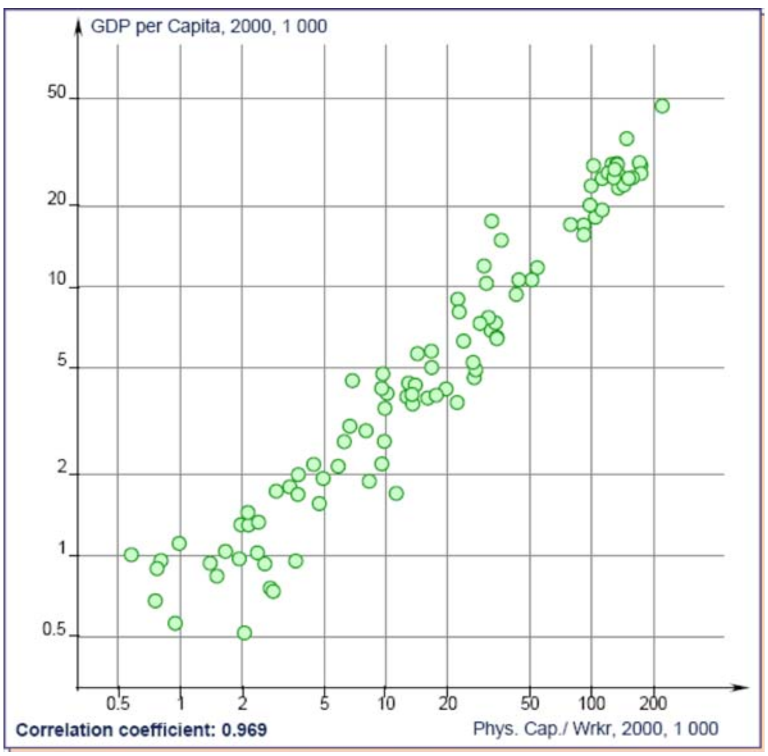
San Francisco State University Michael Bar

Growth

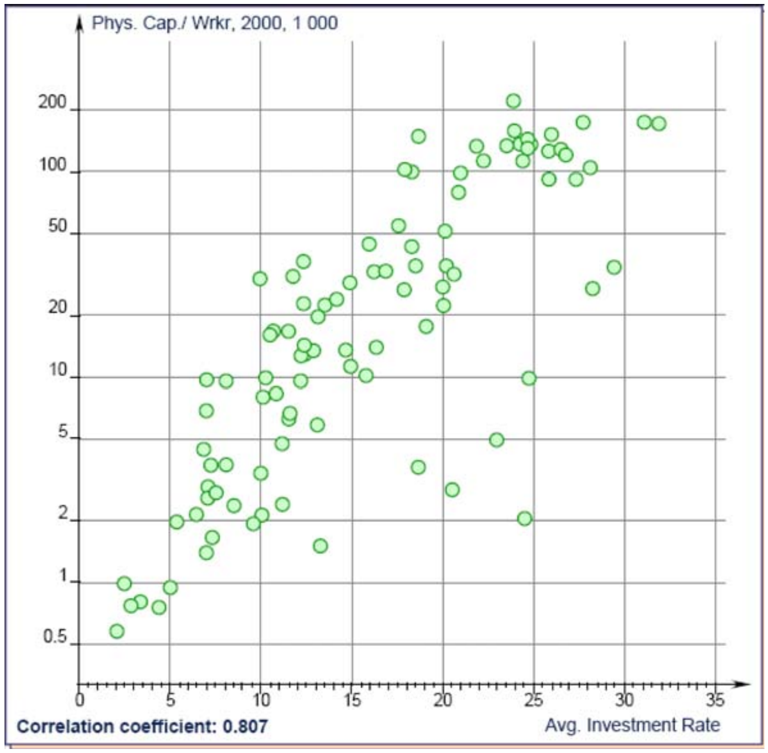
1 Introduction

The most important macroeconomic observation in the world is the huge differences in output (and income) per capita across countries. More than 60% of the world population is at least 7 times poorer than the average American. In the poorest countries the GDP per capita is at least 40 times smaller than that in the U.S. What accounts for such big differences? We will not attempt to answer this question here, but rather focus on one important source of cross country differences - the difference in capital per worker.

The next figure shows the scattergram of capital per worker and GDP/capita in a large sample of countries. We see that there is a strong positive correlation between capital per worker and GDP/capita across countries. In other words, countries with high GDP/capita tend to have higher capital per worker.



But why some countries have much less capital per worker than others? To answer this question recall that capital is created through investment in new capital. The next graph shows the average investment over 1960-2000 v.s. physical capital per worker in 2000. Observe that, not surprisingly, there is a strong positive correlation between the investment rate and physical capital per worker.



These observations motivated the Solow growth model. Robert Solow received a Nobel Prize in Economics in 1987 "for his contributions to the theory of economic growth". Here is part of the press release which describes Solow's contribution:

"*The study of the factors which permit production growth and increased welfare has been a central feature in economic research for many years. Robert M. Solow's prize recognizes his exceptional contributions in this area.*

It is eminently reasonable to imagine that increased per capita production in a country may be the result of more machines and more factories (a greater stock of real capital). But this increased production may also be due to improved machines and more efficient production methods (which may be termed technical development). In addition, better education and training, and improved methods of organizing production may also give rise to increased productivity. The discovery of fresh natural resources, or improvements in a country's position on the world market, may also lead to higher standards of living. Solow has created a theoretical framework which can be used in discussing the factors which lie behind economic growth in both quantitative and theoretical terms. This framework can also be exploited to measure empirically the contributions made by various production factors in economic growth."

2 The Solow Model

2.1 Description of the model

- Output is produce according to $Y_t = A_t K_t^{\theta} L_t^{1-\theta}, \; 0 < \theta < 1$.

- Capital evolves according to $K_{t+1} = K_t (1 - \delta) + I_t$, where δ is the depreciation rate and I_t is aggregate investment.
- People save a fraction s of their income. This fraction is exogenous[1]. Thus, the total saving and total investment in this (closed) economy is

$$S_t = I_t = sY_t$$

- The population of workers grows at a constant rate of n, which is exogenous in this model. Thus, $L_{t+1} = (1 + n) L_t$.

We neglect the differences between population and population of workers for the sake of simplicity, and use the terms "output per worker" and "output per capita" Interchangeably.

2.2 Working with the model

Now we derive the predictions of the model. The output per worker is:

$$y_t = \frac{Y_t}{L_t} = \frac{A_t K_t^{\theta} L_t^{1-\theta}}{L_t} = A_t \left(\frac{K_t}{L_t} \right)^{\theta} = A_t k_t^{\theta}$$

The law of motion of capital per worker is

$$\begin{aligned} \frac{K_{t+1}}{L_{t+1}} &= \frac{K_t (1 - \delta)}{L_{t+1}} + \frac{I_t}{L_{t+1}} \\ k_{t+1} &= \frac{K_t (1 - \delta)}{L_t (1 + n)} + \frac{sY_t}{L_t (1 + n)} \end{aligned}$$

$$k_{t+1} = \frac{k_t (1 - \delta)}{1 + n} + \frac{sA_t k_t^{\theta}}{1 + n} \qquad (1)$$

Equation (1) describes the law of motion of physical capital per worker. If A_t is fixed at some level A, then the law of motion can be illustrated graphically, as in figure (1).

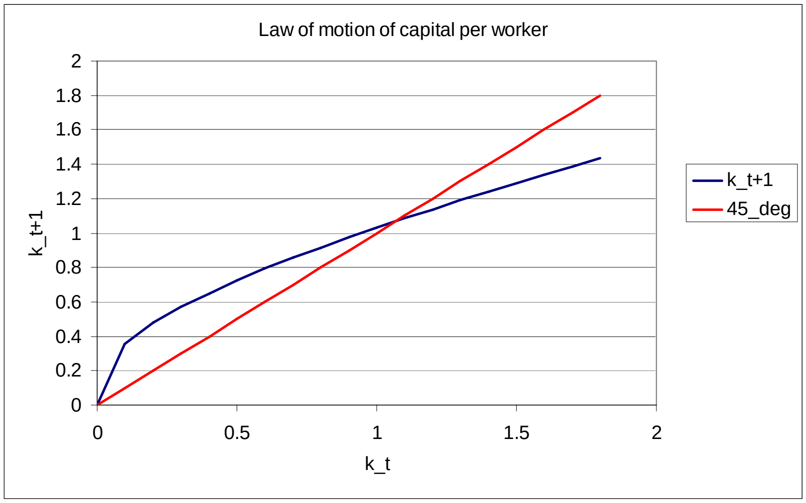
With fixed productivity it can be shown that the capital per worker converges to a steady state level, such that

$$k_{t+1} = k_t = k^{ss} \quad \forall t$$

The steady state level of capital per worker can be seen in the graph at the intersection of the law of motion equation with the 45^0 line. It can be shown that starting from any level of capital per worker, it converges to the steady state level k^{ss}. Thus, the prediction of the Solow model is that with fixed A, the capital per worker will converge to k^{ss}.

[1] We call a variable **endogenous** if it is determined within the model and **exhogenous** if it is determined outside the model. For example, in the model of a market (supply and demand diagram), the price and quantity traded of the good are endogenous variables, while other variables that determine the location of the supply and demand curve, such as income and prices of other goods, are assumed exogenous.

Figure 1: Law of motion of physical capital per worker.



2.2.1 Finding the steady state

Using the law of motion and the definition of the steady state

$$
\begin{aligned}
k_{t+1} &= \frac{k_t\left(1-\delta\right)}{1+n} + \frac{sAk_t^\theta}{1+n} \\
k &= \frac{k\left(1-\delta\right)}{1+n} + \frac{sAk^\theta}{1+n} \\
k\left(1+n\right) &= k\left(1-\delta\right) + sAk^\theta \\
k\left(n+\delta\right) &= sAk^\theta \qquad (2)
\end{aligned}
$$

The intuition behind the last equation is as follows. The left hand side shows the decline in capital per worker due to depreciation and growth in the number of workers. The right hand side is the investment per worker, i.e. the increase in capital per worker. At the steady state the decline in capital per worker due to depreciation and growth in the labor force must be offset by the increase in capital per worker due to investment.

The **steady state capital per worker** is

$$k_{ss} = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\theta}} \qquad (3)$$

The **steady state output per worker** is

$$y_{ss} = Ak_{ss}^\theta = A\left(\frac{sA}{n+\delta}\right)^{\frac{\theta}{1-\theta}} = A^{\frac{1}{1-\theta}}\left(\frac{s}{n+\delta}\right)^{\frac{\theta}{1-\theta}} \qquad (4)$$

The **steady state consumption per worker** is

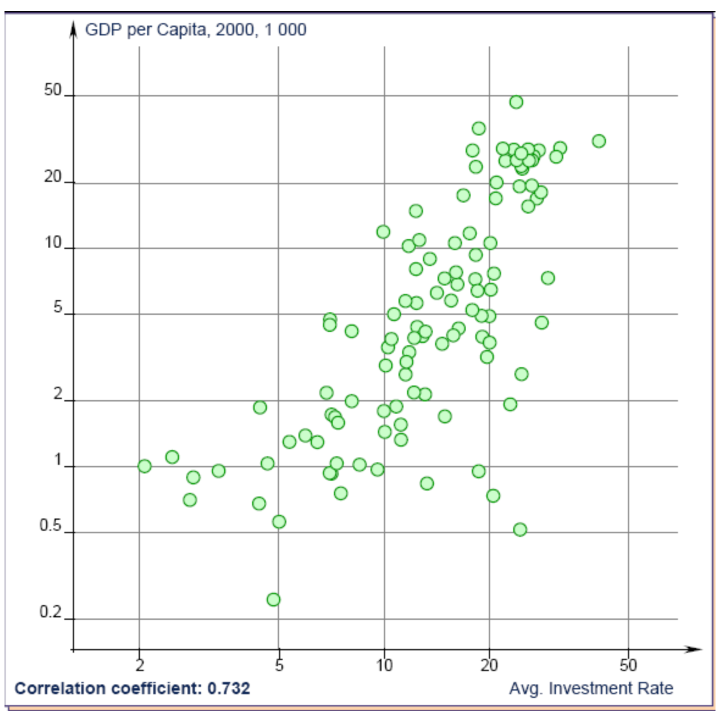
$$c_{ss} = \left(1-s\right)Ak_{ss}^\theta \qquad (5)$$

2.2.2 The predictions of the model

Observe that k_{ss} is increasing in the saving (=investment) rate and productivity, and decreasing in the population growth rate and depreciation. This result is fairly intuitive. First, higher saving rate means that investment per worker is higher, and thus capital per worker should be higher. Higher productivity means that investment per worker is higher simply because more output is produced per worker. This also leads to higher capital per worker in the steady state. Equation (2) can help with developing the intuition. Think of the right hand side of (2) as the "flow in" the stock of capital per worker, i.e. the investment per worker. Higher s or higher A increase the "flow in", and result in higher stock of capital per worker. Now from equation (3) we can see that steady state capital per worker is decreasing in depreciation and the growth rate of population. Higher depreciation means that the "flow out" of the stock of capital is higher, so the stock of capital is lower. Similarly, higher growth rate of the workers population also reduces the capital available per worker, and in a sense works just like depreciation. Observe that n and δ appear together in equations (3) and (4). Similarly, the steady state output per worker y_{ss} is increasing in the saving rate and productivity, and decreasing in the population growth rate and depreciation, just like k_{ss}. This is just because output per worker is increasing function of the capital per worker.

The Solow model therefore predicts that countries with higher investment rates, should on average, enjoy higher standard of living. This prediction is consistent with the data, as can be seen from figure (2).

Figure 2: Investment rate and GDP per capita.



The Solow model also predicts that countries with higher growth rate of population, should on average enjoy lower standard of living. This prediction is also consistent with the data.

According to the Solow model, higher saving rate leads to higher output per worker. Does this mean that the policy recommendation implied by the model is to save as much as possible? The answer to this question is absolutely no. Suppose that at the extreme the consumers save all their income. If this happens, the consumers will not consume anything, and just starve and die. The next section discusses the optimal saving rate, i.e. the saving rate that maximizes the steady state consumption per capita.

2.2.3 Optimal saving rate

Notice that although higher saving rate leads to higher steady state level of capital per worker and output per worker, it does not necessary lead to higher consumption per worker. Observe from equation (5) that on the one hand higher s leads to higher income per worker, but on the other hand higher saving rate means that a smaller fraction of that income is consumed. We can find the optimal saving rate, i.e. the saving rate that maximizes the steady state consumption per worker. This saving rate is called the **golden rule saving rate**.

$$c_{ss} = (1-s)\,Ak_{ss}^{\theta} = Ak_{ss}^{\theta} - (n+\delta)\,k_{ss}$$

$$\max_{k_{ss}} c_{ss} = Ak_{ss}^{\theta} - (n+\delta)\,k_{ss}$$

First order condition:

$$\begin{aligned} \theta A k_{GR}^{\theta-1} &= n+\delta \\ k_{GR} &= \left(\frac{\theta A}{n+\delta}\right)^{\frac{1}{1-\theta}} \end{aligned}$$

Now comparing this with the steady state capital

$$k_{ss} = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\theta}}$$

implies that

$$s_{GR} = \theta$$