San Francisco State University Michael Bar

Growth Rates

Definition: **Time series** - the values of a variable recorded at different points in time constitutes a time series.

Time series is collected by a number of different agencies in the economy. For example the Bureau of Economic Analysis (BEA) collects data about National Income and Product. Federal Reserve System collects data on Monetary aggregates and Interest rates. The Bureau of Labor Statistics collects data on Employment and wages. Data are also measured in different time intervals, so we have annual data , which is recorded once a year, quarterly data recorded four times a year. We also have data recorded every minute such as stock prices on the NYSE.

1 Measuring Rates of change.

We distinguish between 2 types of variables. *Discrete time variable* is a variable that we can measure only countable times per year. GDP is an example of such variable, it is measured 4 times a year. *Continuous time variable* is a variable that can be measured at any instant. For example, the temperature in Minneapolis can be measured continuously. It is important to distinguish between the nature of the variable and our ability to measure it. The GDP is continuous by nature since every instant something is being produced. However, we are not able to measure the GDP at the time it is produced. The BEA can estimate the output only after it was sold to the buyers. Therefore, we are going to treat economic variables such as GDP as discrete variables.

1.1 Discrete time variables

Notations:

Let y_t = value a variable at time t.

y_{t+1} = value of GDP at time $t+1$.

The rate of change in y from period t to $t+1$ is given by

$$\frac{y_{t+1} - y_t}{y_t} \tag{1}$$

Example: Suppose the price of a good was \$76 at 2000 and \$87 at 2001. What is the rate of change in the price from 2000 to 2001?

Solution:

$$\frac{p_{t+1} - p_t}{p_t} = \frac{87 - 76}{76} = 0.14474 = 14.474\%$$

Consider the special case when the rate of growth is a constant over time, say g. That is $\frac{y_{t+1}-y_t}{y_t} = g$ for all values of t. This implies that $y_{t+1} = (1+g)y_t$. Hence, we can express the value of y at time t in the following way

$$y_t = (1+g)^t y_0 \tag{2}$$

where y_o = Initial value y at time $t = 0$.

Example: Suppose that you invest \$1000 in a trust fund that promises 5% annual interest rate. How much money will you have in the fund after 10 years?

Solution:

$$y_t = (1 + 0.05)^{10} 1000 = \$1628.9$$

Example: US GDP per capita grows at constant rate of 2% per year. After how many years will it double?

Solution:

$$\begin{aligned}
2y_0 &= (1 + 0.02)^t y_0 \\
2 &= (1.02)^t \\
\ln(2) &= t \ln(1.02) \\
t &= \frac{\ln(2)}{\ln(1.02)} \approx 35
\end{aligned}$$

Example: Korea grows at 4%. How long would it take for Korea to catch up with the US if the US GDP grows at a constant rate of 2% per annum and the Korean GDP is just half the size of US GDP?

Solution:

$$\begin{aligned}
(1 + 0.02)^t 2y_o &= (1 + 0.04)^t y_o \\
(1 + 0.02)^t 2 &= (1 + 0.04)^t \\
t \ln(1.02) + \ln(2) &= t \ln(1.04) \\
t &= \frac{\ln(2)}{\ln(1.04) - \ln(1.02)} \approx 35.7
\end{aligned}$$

1.2 Continuous time variables

Now we assume that the variable y is a differentiable function of time, $y(t)$. This implies that it is continuous function. The following formula gives the rate of change of a continuous variable. This is the continuous time analog to formula 1.

$$\frac{d \ln(y(t))}{dt} \tag{3}$$

To show why this gives the rate of change, use the chain rule to get

$$\frac{d \ln(y(t))}{dt} = \frac{1}{y(t)} \frac{dy(t)}{dt}$$

The term $\frac{dy(t)}{dt}$ (or in Newton's notation $\dot{y}$) gives the change in y per "small" unit of time and it is analogous to the numerator in equation 1, $y_{t+1} - y_t$. The denominator in both formulas is the same.

Example: suppose that the population of fish at time t is given by $y(t) = 0.01t$. Find the rate of growth of the fish population at time $t = 7$ and $t = 8$.

Solution:

$$\frac{d\ln(y(t))}{dt} = \frac{d\ln(0.01t)}{dt} = \frac{d[\ln(0.01)+\ln(t)]}{dt} = \frac{1}{t}$$

Hence, after 7 periods the growth rate is $\frac{1}{7}$ and after 8 years it is $\frac{1}{8}$ (we have diminishing growth rate).

Example: Same as before, but now the population at time t is $y(t) = e^{0.05t}y_0$.

Solution:

$$\frac{d\ln(y(t))}{dt} = \frac{d[0.05t+\ln(y_0)]}{dt} = 0.05 = 5\%$$

Here we got constant growth rate. This example leads us to the continuous time analog to formula 2. This formula gives the value of y at time t under the constant growth rate assumption

$$y(t) = e^{gt}y(0) \tag{4}$$

where g is the constant growth rate.

2 Rate of change of a product and ratio

There are two important approximations for the growth rate of a *product of two variables* and for the growth rate of *ratio of two variables*. Let a "hat" on top of the variable denote its rate of change, i.e., $\widehat{x} = \frac{x_{t+1}-x_t}{x_t} = \frac{x_{t+1}}{x_t} - 1$. Then the two rules are:

1. The growth rate of a product is approximately the sum of the growth rates, i.e.

$$\widehat{xy} \approx \hat{x} + \hat{y}$$

2. The growth rate of the ratio is approximately the difference of the growth rates

$$\widehat{\left(\frac{x}{y}\right)} \approx \hat{x} - \hat{y}$$

Proof. 1. From the definition of growth rate, we have

$$1 + \widehat{xy} = \frac{x_{t+1}y_{t+1}}{x_t y_t}$$

Taking ln of both sides

$$\ln\left(1+\widehat{xy}\right) = \ln\left(\frac{x_{t+1}y_{t+1}}{x_t y_t}\right) = \ln\left(\frac{x_{t+1}}{x_t}\right) + \ln\left(\frac{y_{t+1}}{y_t}\right) = \ln\left(1+\hat{x}\right) + \ln\left(1+\hat{y}\right)$$

Recall that $\ln(1+g) \approx g$ for small g. Thus, the above equation is approximately

$$\widehat{xy} = \hat{x} + \hat{y}$$

2. From the definition of growth rate, we have

$$1 + \widehat{\left(\frac{x}{y}\right)} = \left(\frac{x_{t+1}}{y_{t+1}}\right) / \left(\frac{x_t}{y_t}\right) = \left(\frac{x_{t+1}}{y_{t+1}}\right) \cdot \left(\frac{y_t}{x_t}\right) = \left(\frac{x_{t+1}}{x_t}\right) / \left(\frac{y_{t+1}}{y_t}\right)$$

Taking ln of both sides

$$\ln\left(1+\widehat{\left(\frac{x}{y}\right)}\right)=\ln\left(\frac{x_{t+1}}{x_t}\right)-\ln\left(\frac{y_{t+1}}{y_t}\right)=\ln\left(1+\hat{x}\right)-\ln\left(1+\hat{y}\right)$$

Recall that $\ln(1+g)\approx g$ for small g. Thus, the above equation is approximately

$$\widehat{\left(\frac{x}{y}\right)}=\hat{x}-\hat{y}$$

■

Remark: The above can be proved without resorting to the logarithms.

Proof. 1. From the definition of growth rate, we have

$$1+\widehat{xy}=\frac{x_{t+1}y_{t+1}}{x_t y_t}=\left(1+\hat{x}\right)\left(1+\hat{y}\right)=1+\hat{x}+\hat{y}+\hat{x}\hat{y}$$

For small growth rates, the product $\hat{x}\hat{y}$ is negligible (e.g. $2\%\cdot 3\%=0.0006$), so we have

$$\widehat{xy}\approx\hat{x}+\hat{y}$$

2. Omitted. ■

2.1 Examples

1. Suppose that during the last year, the price of a product increased by 2% and the quantity sold increases by 1.5%. What is the approximate growth rate of the revenue?

   Solution:

   $$\widehat{P\cdot Q}\approx\hat{P}+\hat{Q}=2\%+1.5\%=3\%$$

   Remark: If we wanted the exact growth rate of the revenue, then letting the initial values of price and quantity be P_0 and Q_0 we have

   $$1+\widehat{P\cdot Q}=\frac{\overbrace{(1+0.02)\,P}^{\text{new price}}\cdot\overbrace{(1+0.01)\,Q}^{\text{new quantity}}}{P\cdot Q}=(1+0.02)\,(1+0.01)=1.0302$$

   which is close to the approximation.

2. Suppose that during the last year, the real GDP is grew at 2.5% and population grew at 1%. What is the approximate growth rate of GDP per capita?

   Solution:

   $$\widehat{\left(\frac{Y}{N}\right)}\approx\hat{Y}-\hat{N}=2.5\%-1\%=1.5\%$$

   Remark: if we wanted the exact growth rate, then letting the initial levels of GDP and population be Y_0 and N_0, we have

Proof.

$$1+\widehat{\left(\frac{Y}{N}\right)}=\overbrace{\left(\frac{(1+0.025)\,Y_0}{(1+0.01)\,N_0}\right)}^{\text{new GDP per capita}}/\left(\frac{Y_0}{N_0}\right)=\frac{1+0.025}{1+0.01}=1.014851485$$

which is close to the approximation. ■

3 Logarithmic scale

Suppose that a variable y grows at constant rate g and initial value of y_0. Then the value of y at time t is given by

$$y_t = y_0(1+g)^t$$

Now, if we take the natural logarithm of y_t, we get that $\ln(y_t)$ is a linear function of time:

$$\ln(y_t) = \ln(y_0) + t\ln(1+g)$$

This is a linear function of time, with slope of $\ln(1+g)$ and intercept $\ln(y_0)$. Now we can show that the slope of this function is approximately equal to the growth rate, for small g, i.e. $\ln(1+g) \approx g$

Proof. We need to prove that

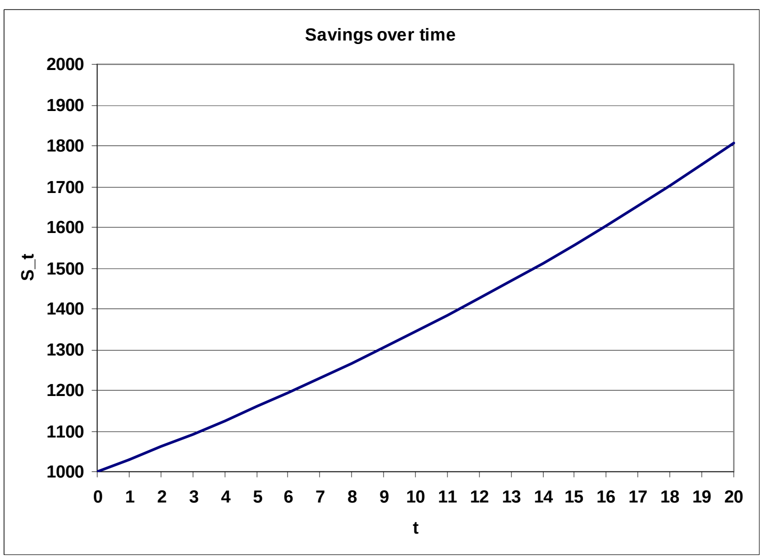
$$\lim_{g\to 0} \frac{\ln(1+g)}{g} = 1$$

Notice that when $g \to 0$, both the numerator and the denominator in the limit go to zero. In other words, we have a limit of the form of $\frac{0}{0}$. Using L'Hopital's rule we get

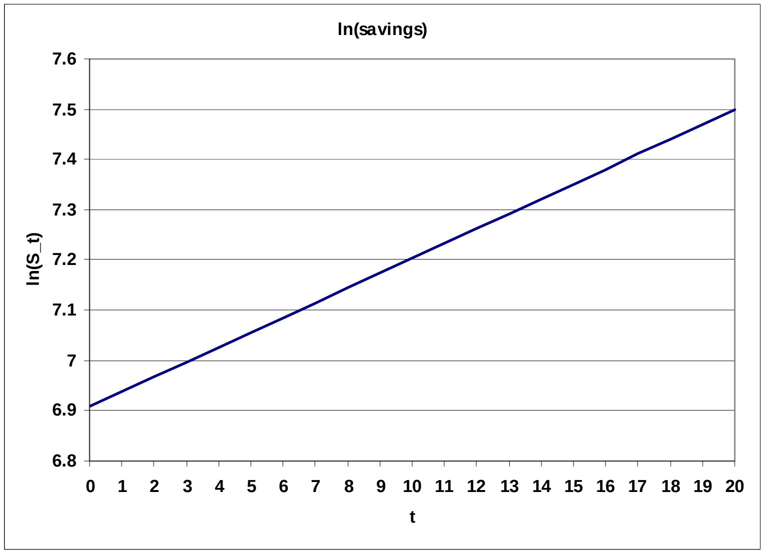
$$\lim_{g\to 0} \frac{\ln(1+g)}{g} = \lim_{g\to 0} \frac{1/(1+g)}{1} = 1$$

■

Example. Suppose that you deposit \$1000 in a savings account, with interest rate of 3%. The amount of money you have in the in the savings account at any time t is $s_t = 1000 \cdot (1+0.03)^t$, and is shown in the next graph.



Now the ln of savings is $\ln(s_t) = \ln(1000) + t\ln(1+0.03)$ and shown in the next graph.



Notice that ln of savings is a linear function of time. Also observe that the slope of $\ln(s_t)$ is approximately equal to

$$\approx \frac{7.5 - 6.9}{20} = \frac{0.6}{20} = 3\%$$

The properties of logarithmic scale are very useful every time we look at data that is growing over time. By looking at the original data we cannot tell whether it is growing at constant rate or not. But when we plot the ln of the variable, we can see right away if the variable is growing at constant rate or not. That is, if the ln of the variable looks like linear function of time, we conclude that the original variable is growing at constant rate. Moreover, we can immediately compute the approximate growth rate of the original variable from the slope of the ln, as shown in the previous example.