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Saving and Investment

The process of economic growth depends, among other things, on the ability of firms to expand their productive capacity through investment in additional equipment. Firms can finance their investment from retained earnings (also called undistributed profits or business saving) or borrow funds from households who save. In this chapter we discuss the relationship between saving and investment in the macroeconomy, and present a theory of saving and investment. We will examine what factors affect investment decisions by the firms and saving decisions by the households, and how those decisions are affected by government policies.

Before we start the formal discussion of saving and investment, we need to introduce two general concepts. A **stock variable** is a magnitude measured at a point in time (say at the end of the year), and a **flow variable** is a variable measured over a given time interval (say over the year). For example, the stock of capital in the U.S. on December 31^{st} 2005, is a stock variable. Investment that took place in 2005 is a flow variable. As another example, the saving during 2005 is a flow variable and the **savings** at the end of 2005 (the value of all the balances of saving accounts) is a stock variable. The flow variables determine the value of the stock variable at the end

1 Saving and Investment Equation

In any economy there exists an accounting identity that relates saving and investment. In this section we derive this relationship, called the **saving and investment equation**. The GDP is given by

$$GDP = C + G + I + NX \tag{1}$$

where C is personal consumption expenditure, G is government consumption expenditure, I is gross domestic investment, and $NX = X - IM$ is net exports (exports minus imports). We define disposable income as

$$YD = GDP + TR - T \tag{2}$$

where TR are transfer payments by the government (such as unemployment insurance benefits, social security, medicare,...). and T is taxes. We define the private saving as

$$S_P = YD - C \tag{3}$$

That is, the private saving is the disposable income that is not consumed. Similarly, the government saving is

$$S_G = T - TR - G \tag{4}$$

which is the government income that is not spent on government consumption or transfer payments. Government deficit is then

$$Def = -S_G = G + TR - T \tag{5}$$

Now add TR and subtruct T from equation (1)

$$GDP + TR - T = C + G + TR - T + I + NX$$

Now using the definition of disposable income in equation (2)

$$YD = C + Def + I + NX$$

Finally, using the definition of the government deficit in equation (5) gives the **saving and investment equation**

$$S = S_P + S_G = I + NX \tag{6}$$

The left hand side of (6) is the gross national saving S, which is the sum of private and government saving. On the right hand side we have the gross domestic investment I and net exports NX, which is also known as **Net Foreign Investment**. In a closed economy we have

$$S_P + S_G = I$$

that is, in a closed economy the total saving is equal to total investment. In an open economy, on the other hand, part of the domestic saving can fund investment abroad, if $S > I$. It is also possible in an open economy that the domestic saving are insufficient to fund all of the domestic investment, if $S < I$, and in this case part of the domestic investment is funded by foreigners. If $NX > 0$, then the economy is exporting more goods and services than what it imports, i.e. the country is experiencing a trade surplus. This means that the domestic economy is accumulating foreign assets, since the rest of the world has to borrow from the domestic economy. If on the other hand, $NX < 0$, this means that the economy is importing more goods and services than its exports to the rest of the world, i.e. the country is experiencing trade deficit (trade deficit is defined as $-NX$). In this case the domestic economy has to borrow from the rest of the world and the rest of the world is accumulating domestic assets. Thus, in the open economy, total saving is equal to the domestic investment plus net foreign investment, as equation (6) states

What can we learn from the saving and investment equation (6)?

1. The domestic investment can be financed by domestic saving and by foreign saving. Rewriting (6) gives

   $$I = S_P + S_G - NX$$

   The term $-NX$ represents the investment of the rest of the world in the domestic economy. For example, if $I = 20$, $S_P + S_G = 15$ and $NX = -5$, then

   $$\underbrace{I}_{20} = \underbrace{S_P + S_G}_{15} - \underbrace{NX}_{-5}$$

   which means that part of the domestic investment is financed by domestic saving (15) and part is financed by foreign saving (5).

   If on the other hand we have $I = 20$, $S_P + S_G = 25$ and $NX = 5$, then

   $$\underbrace{I}_{20} = \underbrace{S_P + S_G}_{25} - \underbrace{NX}_{5}$$

   which means that the domestic saving finances not only the domestic investment, but also finances some of the investment in the rest of the world.

2. The government can finance its deficit in two ways: (1) borrowing from domestic residents or (2) borrow from the rest of the world. Rewriting equation (6) gives

$$\begin{aligned} S_P + S_G &= I + NX \\ Def &= S_P - I - NX \end{aligned}$$

Thus, we can see that an increase in government deficit has to be associated with either increase in private saving, or decrease in gross domestic investment or an increase in the trade deficit (borrowing from abroad).

2 Saving and Investment in the U.S.

Lets take a look at the behavior of total investment in the U.S. and how it was financed. Figure (1) shows the total investment in the U.S. as a fraction of GDP.We see that the total

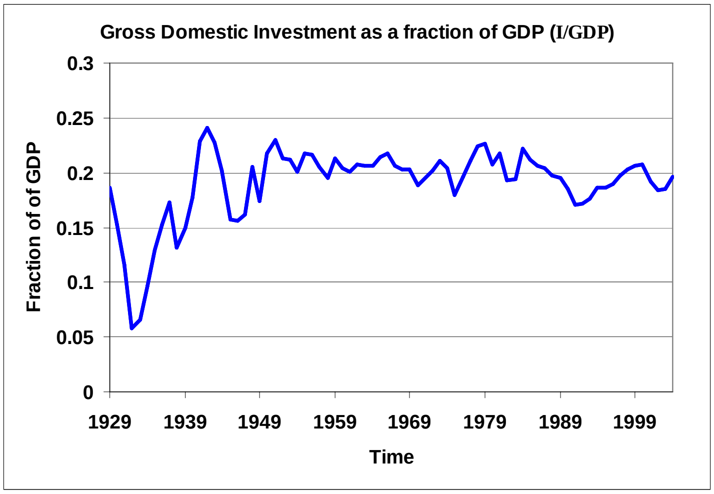


Figure 1: Gross Domestic Investment as a fraction of GDP.

investment since 1929 tends to fluctuate around 20% of GDP. In other words, the investment rate in the U.S. is about 20% of GDP, and this rate does not change much over time. From equation (6) we know that domestic investment can be financed by domestic saving (private and government), or carried out by foreigners. That is, $I = S_P + S_G - NX$. Lets examine the funding sources of investment. Figure (2) shows the private saving as a fraction of GDP.

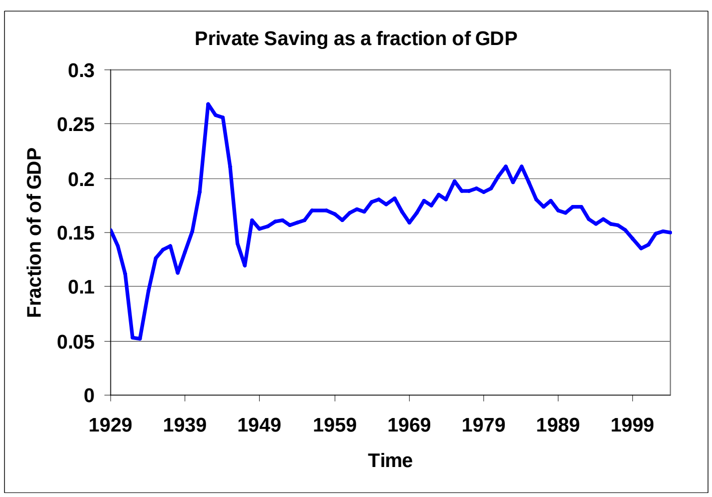


Figure 2: Private Saving as a fraction of GDP.

Notice that in the last data point (in 2004) the private investment is about 15% of GDP. Thus, the private saving is about 75% of the domestic investment. The rest, according to the identity (6) has to come from government saving or from foreign investment. Figure (3) shows the government saving as a fraction of GDP.Notice that recently the government is

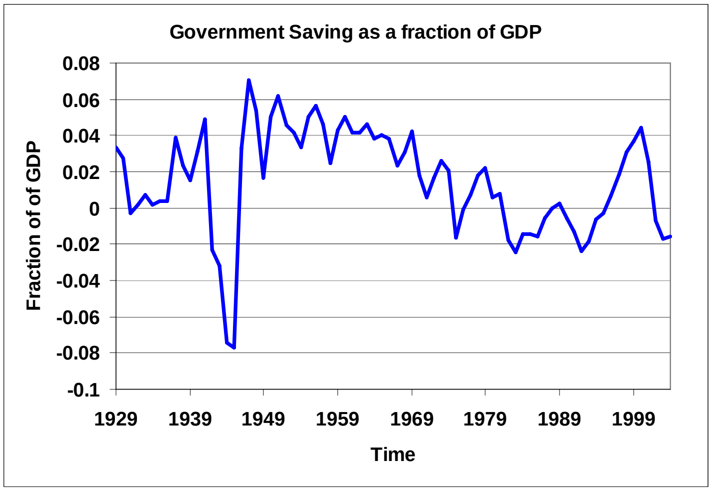


Figure 3: Government saving as a fraction of GDP.

running a budget deficit, and in 2004 the deficit was about 2% of GDP. Thus, the government does not "help" to finance the domestic investment.

The rest of the funding for the domestic investment has to come from abroad. Figure (4) shows the net exports as a fraction of GDP.We can see that in the last 30 years the U.S. is

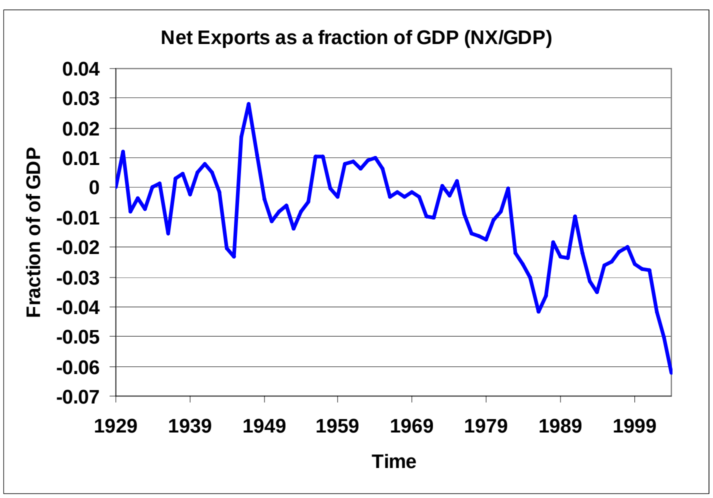


Figure 4: Net Exports as a fraction of GDP.

experiencing trade deficit. This means that foreigners accumulate U.S. assets. In particular, in 2004 the trade deficit was about 6% of GDP, which corresponds to foreign investment in the U.S.

3 Intertemporal Choice Model (Saving Theory).

In the first section of these notes we showed the relationship between saving and investment in the economy, called the **saving and investment equation**:

$$S = I + NX$$

Our next goal is to investigate the determinants of saving and investment. In this section we build a model in which consumers make explicit decisions about consumption and saving. Before presenting the model, let's take a moment to think about what factors might affect the saving decision of households. We point out three factors that might be important determinates of saving.

Why do people save? Saving is a process of giving up current consumption in order to increase the future consumption. Therefore, we expect that our saving decisions would depend on our current and future income. Typically, individuals who work and expect a decrease in their income when they retire, tend to save some of their current income for retirement. In contrast, other individuals who expect an increase in their future income, tend to borrow (have negative saving). For example, many students take student loans while they are studying, and plan to repay the loan when they graduate and earn higher income. Therefore, current and future income, are among the most important factors that affect the saving decisions.

Another important factor that affects the saving decision is the interest rate. If an individual gives up some of his current consumption and decides to save, he will be able to increase his future consumption. But the question is by how much? The **real** interest rate gives the answer to that question. If you walk into a bank and open a savings account, the interest rate that the bank will offer you is a **nominal** interest rate. The nominal interest rate tells you how many extra **dollars** you will get in the next period when you save one dollar today. For example, if the annual nominal interest is 10%, this means that when you deposit \$1 today, you will receive your \$1 back, plus \$0.1 interest.

What consumers care about though is not how much money they got, but how much consumption they can buy with their money. Suppose that as before, the nominal interest rate is 10%. In addition, suppose that there is some consumption good, say burger, which currently costs 1\$. What you really care about is how many extra burgers you will get next year when you give up one burger this year. Suppose that the inflation rate is 5%, so that the price of a burger next year is \$1.05. Now, when you give up one burger today, and save the \$1 in the savings account, you will receive \$1.1 in the next year, and with this money you can buy $1.1/1.05 \approx 1.048$ burgers. Thus, when you give up one burger today, you get 0.048 extra burgers in the future. We therefore say that the **real interest rate** is 0.048 or 4.8%.

Formally, the **real interest rate** is the extra amount of consumption that one gets in the future when he gives up one unit of current consumption. In contrast, the **nominal interest rate** is the extra dollars that one gets in the future when he gives up one dollar today. Let i be the nominal interest rate, r be the real interest rate, and π be the inflation rate. Then the relationship between the nominal interest rate and the real interest rate is given by:

$$\boxed{\frac{1+i}{1+\pi} = 1+r}$$

If the nominal interest rate and the inflation rates are small, then we can derive an approximation formula to the above. Taking ln from both sides gives

$$\ln(1+i) - \ln(1+\pi) = \ln(1+r)$$

If i, r, π are small, the above is **approximately**

$$\boxed{r = i - \pi}$$

Thus, the real interest rate is approximately equal to the nominal interest rate minus the inflation rate. In the burger example, the nominal interest rate 10% and the inflation rate is 5%, and then the real interest rate is approximately 5%. Recall that the exact real interest rate was 4.8%, which is close to 5%.

To summarize this discussion, since real interest rate determines how much extra future consumption we expect to get when we save one unit of current consumption, then we suspect that the real interest rate would be one of the pivotal factors that affect our saving decisions.

The third factor that determines saving is our preferences. An individual who values current consumption a lot and does not value future consumption much (someone who "lives the day"), will tend to save little or even borrow. On the other hand, someone who values future consumption or consumption of his children and grand children a lot will tend to save more.

The three factors that affect the saving decision are therefore, preferences, current and future income, and the real interest rate.

3.1 The Model

Consumers: There are *N* identical consumers that live for two periods (1 and 2) and derive utility from consumption c_1 and c_2 in the two periods: $U(c_1, c_2)$. Consumers receive income y_1 and y_2 in the two periods and pay a lump sum tax t_1 and t_2 to the government. The consumers decide how much to consume in each period and how much to save in the first period. We denote the saving in the first period by *s*. Consumers can borrow and lend at **real** interest rate *r*, which is assumed exogenously given. Thus the budget constraints in the two periods are

$$\begin{aligned} BC_1: &\quad c_1 + s = y_1 - t_1 \\ BC_2: &\quad c_2 = y_2 - t_2 + (1+r)s \end{aligned}$$

The consumers' problem is therefore

$$\begin{cases} \max\limits_{c_1, c_2, s} U(c_1, c_2) \\ s.t. \\ BC_1: \quad c_1 + s = y_1 - t_1 \\ BC_2: \quad c_2 = y_2 - t_2 + (1+r)s \end{cases}$$

Government: The government collects tax revenues $T_1 = N \cdot t_1$ and $T_2 = N \cdot t_2$ in the two periods and spends G_1 and G_2 in the two periods. The government can borrow and lend at real interest rate *r* with the constraint that the present value of spending = present value of taxes

$$G_1 + \frac{G_2}{1+r} = T_1 + \frac{T_2}{1+r}$$

This means that if the government runs a deficit in the first period, it must borrow the amount of the deficit and pay that amount with the second period's surplus. And if the government has a surplus, it can save the surplus at interest *r* and be able to afford a deficit in the second period. To see this, rearrange the above condition

$$(G_1 - T_1)(1+r) = T_2 - G_2$$

Suppose that the interest rate is $r = 5\%$ and in the first period the government runs a deficit of 100, thus $G_1 - T_1 = 100$. The above condition means that in the second period the government must have a surplus of 105 to pay the debt, i.e. $T_2 - G_2 = 105$.

Now that we have completed the description of the model we would like to analyze the impact on consumers of the following changes:

1 Changes in income: y_1 and y_2
2 Changes in the real interest rate: *r*
3 Changes in government taxes: T_1 and T_2

To answer the above questions we need to solve the consumers' problem. It is convenient to derive the lifetime budget constraint of the consumer. Substitute *s* from the second period budget constraint into the first period's budget constraint. It is easy to do when you divide both sides of BC_2 by $1+r$ to get

$$BC_2: \quad \frac{c_2}{1+r} = \frac{y_2}{1+r} - \frac{t_2}{1+r} + s$$

Now add the two budget constraints and get the **lifetime budget constraint:**

$$\boxed{\underbrace{c_1 + \frac{c_2}{1+r}}_{PV\ of\ lifetime\ \ consumption} = \underbrace{y_1 - t_1 + \frac{y_2 - t_2}{1+r}}_{we=lifetime\ \ wealth}}$$

Thus, the left hand side is the present value of lifetime consumption, and the right hand side is the present value of lifetime net of taxes income, which we call the lifetime wealth (*we*).

The consumers' problem can then be rewritten as

$$\begin{cases} \max\limits_{c_1, c_2} U(c_1, c_2) \\ s.t.\ c_1 + \dfrac{c_2}{1+r} = y_1 - t_1 + \dfrac{y_2 - t_2}{1+r} \end{cases}$$

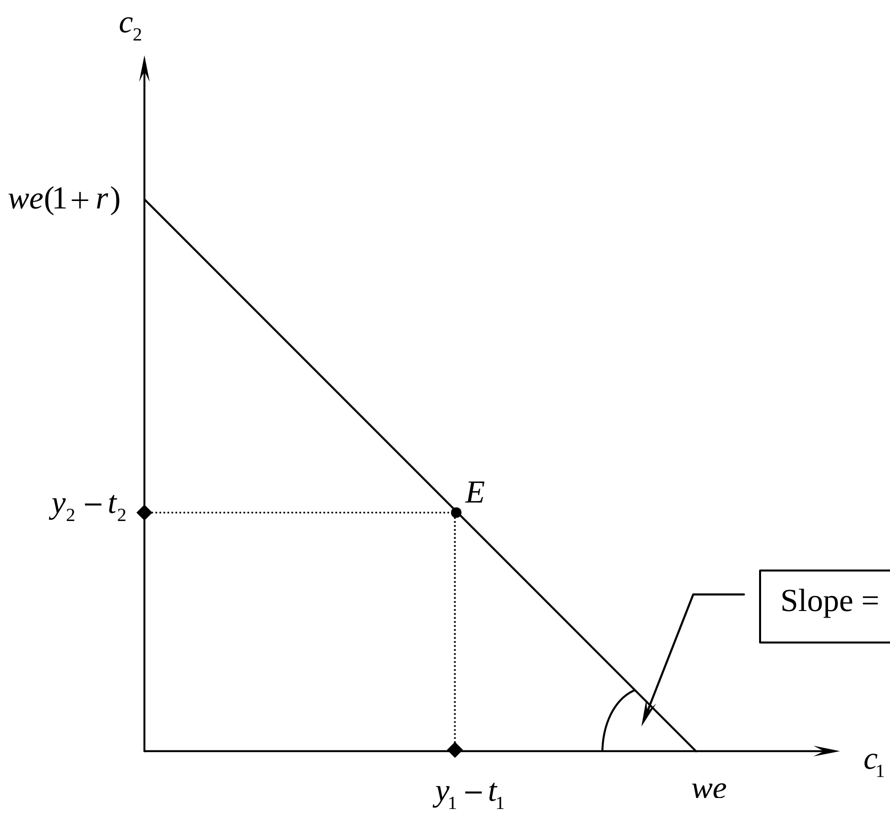
Interpretation: Recall the consumer choice model of choosing the optimal amounts of two goods *x* and *y*, with budget constraint $p_x x + p_y y = I$ (from the micro foundations notes). In that model the slope of the budget constraint is p_x / p_y, i.e., which is the relative price of good *x* in terms of good *y*. Notice that the two period model is very similar to the general model of consumer choice, where the two goods are current consumption and future consumption (c_1 and c_2). We can write the budget constraint concisely as

$$c_1 + \frac{c_2}{1+r} = we,$$

which is similar to

$$p_x x + p_y y = I$$

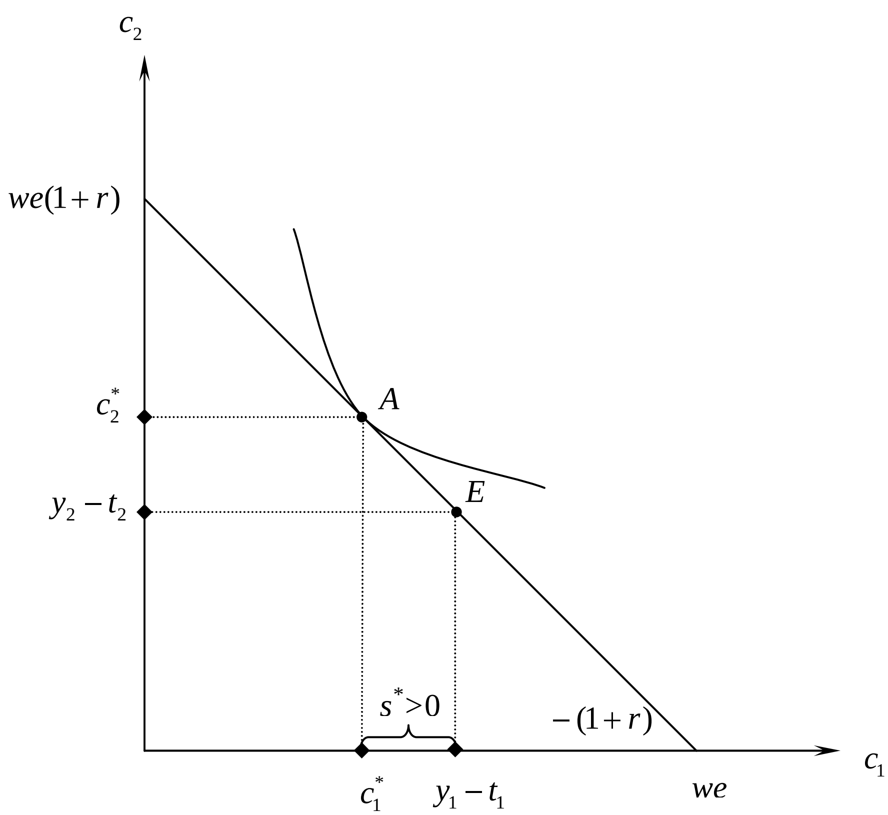
The price of c_1 is 1, and the price of c_2 is $\frac{1}{1+r}$, thus the slope of the budget constraint is $(1+r)$, and it represents the relative price of current consumption in terms of future consumption. Indeed, consider the cost of increasing current consumption by 1 unit. If the consumer saved that unit, then he would have enjoyed an increase of $1+r$ units in the future consumption. Hence the cost of current consumption in terms of future consumption is $1+r$. Thus the consumer can choose the optimal bundle of two goods (c_1 and c_2), given his preferences and given the prices of the two goods. The next figure shows the graph of the lifetime budget constraint.



With free borrowing and lending, it is feasible for this consumer to consume all his wealth in the first period and nothing in the second: ($c_1 = we,\ c_2 = 0$). Similarly, it is feasible for this consumer not to consume anything in the first period and consume all his wealth in the second period: ($c_1 = 0,\ c_2 = we(1+r)$). Also notice that it is feasible for the consumer to consume in each period the income (net of taxes) received in that period: ($c_1 = y_1 - t_1,\ c_2 = y_2 - t_2$). This bundle is denoted by *E* is the consumer's endowment. If the consumer was not allowed to borrow or lend, then he would be forced to consume his endowment, i.e., his net of taxes income in each period. Because the consumers are free to borrow and lend at real interest rate *r*, they can choose other points on the budget constraint. If the consumer chooses a point above *E* on the lifetime budget constraint, then he is a **lender** (his current consumption is less then current income, so he is saving a positive amount). If the consumer chooses a point below *E* on the lifetime budget constraint, then he is a **borrower** (his current consumption is greater than his current income, so he has negative saving).

3.2 Optimal choice

The next figure shows the optimal choice for a consumer who is a lender.



At the optimal bundle (point *A*) we have the usual condition that the marginal rate of substitution between c_1 and c_2 is equal the relative price. That is

$$\boxed{\frac{U_1(c_1,c_2)}{U_2(c_1,c_2)} = 1+r}$$

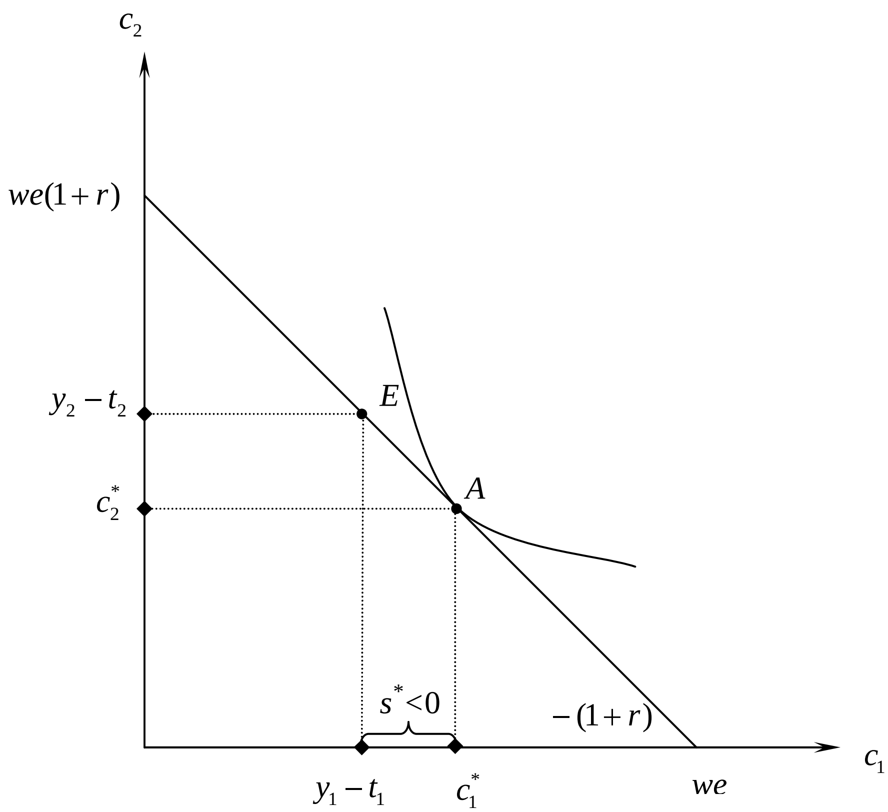
The left hand side is the (absolute value of) the slope of the indifference curves and the right hand side is the (absolute value of) the slope of the budget constraint.

This should look very familiar to you and similar to the optimality condition

$$\frac{U_x(x,y)}{U_y(x,y)} = \frac{p_x}{p_y}$$

Notice that in the case of a lender, the saving is positive.

The next figure shows the optimal choice for a consumer who is a borrower.



Notice that the saving is negative for a borrower.

3.2.1 Changes in income.

In this section we want to analyze the impact of changes in y_1 and y_2 on the consumer's choice (c_1^*, s^*, c_2^*). The consumer's budget constraint is

$$c_1 + \frac{c_2}{1+r} = y_1 - t_1 + \frac{y_2 - t_2}{1+r}$$

a. An increase in current income $y_1 \uparrow$

We see from the budget constraint that an increase in y_1 will shift the budget constraint to the right. If we assume both goods (current consumption and future consumption) are normal, the consumer will increase the consumption in both periods. In order to increase the consumption in the second period the consumer must increase his saving. Thus, an increase in the current income will increase the current consumption by less than the change in the current income. We call this result **consumption smoothing**.

To summarize: $\boxed{y_1 \uparrow \;\Rightarrow\; c_1^* \uparrow, s \uparrow, c_2^* \uparrow,\ \Delta c_1 < \Delta y_1}$

b. An increase in future income $y_2 \uparrow$

We see from the budget constraint that an increase in y_2 will shift the budget constraint to the right. Given that both goods (current consumption and future consumption) are normal, the consumer will increase the consumption in both periods. In order to increase the consumption in the first period the consumer must decrease his saving. Thus, an increase in the future income will increase the future consumption by less than the change in the future income.

To summarize: $\boxed{y_2 \uparrow \;\Rightarrow\; c_1^* \uparrow, s^* \downarrow, c_2^* \uparrow,\ \Delta c_2 < \Delta y_2}$

c. An increase in current and future income $y_1 \uparrow, y_2 \uparrow$.

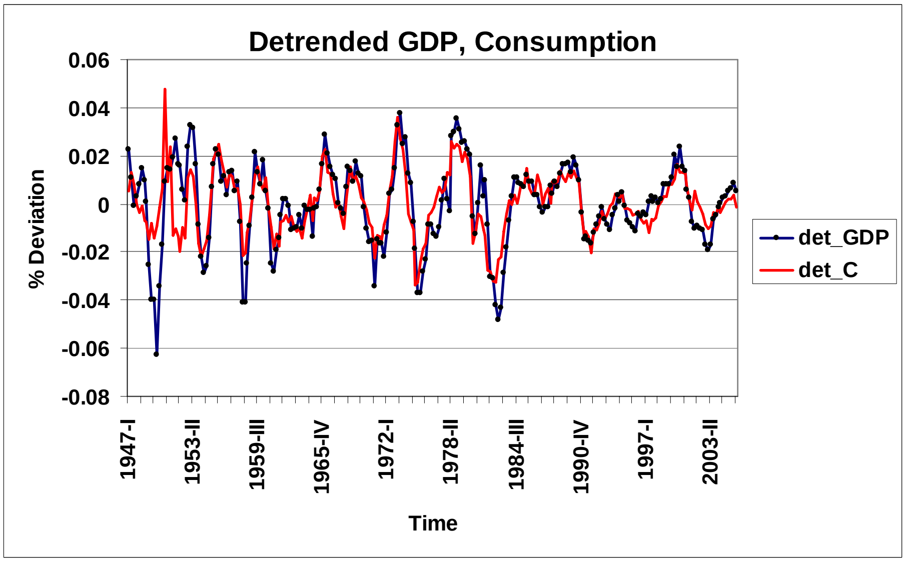
The budget constraint will shift to the right and again consumption in both periods will go up. It is unclear however what will happen to the saving. The impact on saving depends on the relative magnitudes of the changes in y_1 and y_2.

To summarize: $\boxed{y_1 \uparrow, y_2 \uparrow \;\Rightarrow\; c_1^* \uparrow, s^* ?, c_2^* \uparrow}$

Temporary vs. Permanent changes in income

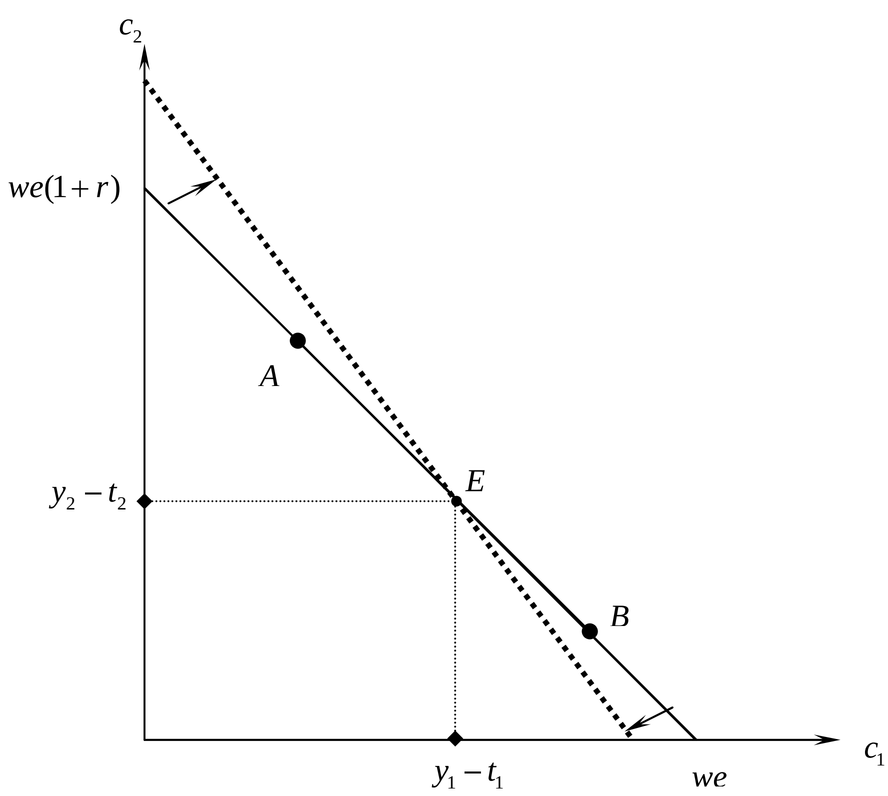
The main point of the above experiments was to show that if the increase in income happens only in one period, then the consumer will increase his consumption in that period by less than the change in that period's income. This is called **consumption smoothing.**

Is there evidence in the data of consumption smoothing? The next figure shows the percentage deviation from trend of real consumption per capita and real GDP per capita in the U.S. What we can see from the next figure is that consumption is smoother than GDP.



3.2.2 Changes in the real interest rate

Suppose that $r\uparrow$. What is the effect of the increase in real interest rate on the budget constraint? First of all, the slope of the budget constraint will increase in absolute value (it will become more steep). Also notice that whatever the interest rate, since the incomes and taxes did not change, the new budget constraint has to pass through the endowment point *E*. Regardless of the interest rate it is always feasible to consume in each period that period's income. The next graph shows the effect of $r\uparrow$ on the budget constraint.



The dashed line is the new budget constraint, after $r\uparrow$.

An increase in the real interest rate has two effects on the consumer. On the one hand the relative price of current consumption in terms of future consumption ($1+r$) has gone up. As a result, the consumer would like to substitute (the now more expensive) current consumption with the (the now cheaper) future consumption. This is called the **substitution effect** – the change in consumption that results from the change in the relative prices. As a result of the substitution effect the consumer will reduce current consumption and increase future consumption. Since current income did not change, the saving must increase. Thus, as a result of the substitution effect we have: $\boxed{c_1^* \downarrow, s^* \uparrow, c_2^* \uparrow}$.

But there is another effect. Notice that if the consumer was lender before the change (for example chose the point *A*), then after the change he can still afford the original bundle, and even bundles that contain more of both goods than *A*. In other words, his purchasing power increased. We call this increase in purchasing power a positive **income effect**. For a borrower (one who consumed at point *B* for example) the opposite happened. After the change he can no longer afford the previous bundle. We call this decrease in the purchasing power a negative **income effect**.

Our definition of income effect is not precise, but it is intuitive and sufficient for the purpose at hand. So for us a positive income effect occurs if at the new prices, even if you take some of the consumer's income he can still afford the old bundle. We say that a negative income effect occurs if at the new prices the old bundle is not affordable.

Assuming that both goods are normal implies that the positive income effect will cause $c_1^* \uparrow, s^* \downarrow, c_2^* \uparrow$ for the lender and $c_1^* \downarrow, s^* \uparrow, c_2^* \downarrow$ for the borrower.

The next table summarizes the results:

	Lender	Borrower
Substitution effect	$c_1^* \downarrow, s^* \uparrow, c_2^* \uparrow$	$c_1^* \downarrow, s^* \uparrow, c_2^* \uparrow$
Income effect	$c_1^* \uparrow, s^* \downarrow, c_2^* \uparrow$	$c_1^* \downarrow, s^* \uparrow, c_2^* \downarrow$
Total effect	$c_1^* ?, s^* ?, c_2^* \uparrow$	$c_1^* \downarrow, s^* \uparrow, c_2^* ?$

3.2.3 Changes in taxes and Ricardian equivalence

We have already analyzed the impact on consumers of the changes in incomes y_1 and y_2. Notice that our analysis also covers the changes in t_1 and t_2 since the budget is

$$c_1 + \frac{c_2}{1+r} = y_1 - t_1 + \frac{y_2 - t_2}{1+r}$$

So an increase in y_1 is just like a decrease in t_1 and an increase in y_2 is just like a decrease in t_2. There is however an important result in public finance – the Ricardian equivalence theorem.

Theorem (Ricardian equivalence):
If the present value of government spending remains unchanged, then changes in the taxes do not affect the households' optimal consumption choice (c_1^*, c_2^*).

Proof:
The government budget constraint is

$$G_1 + \frac{G_2}{1+r} = T_1 + \frac{T_2}{1+r} = N \cdot t_1 + \frac{N \cdot t_2}{1+r}$$

Thus,

$$t_1 + \frac{t_2}{1+r} = \frac{1}{N}\left(G_1 + \frac{G_2}{1+r} \right)$$

We see that any changes in t_1 and t_2 must be such that the present value of taxes that the consumer has to pay remain constant. This means that the consumer's budget constraint remains unchanged,

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} - \left(t_1 + \frac{t_2}{1+r} \right),$$

since the last term on the right hand side (the present value of taxes) is unchanged. This implies that the optimal choice of consumption (c_1^*, c_2^*) for each consumer will remain unchanged.

Notice however that the saving decision of consumers will change, and the aggregate saving can change as well. To see that recall that $s = y_1 - t_1 - c_1$. If the government changes the taxes (t_1 and t_2), then since c_1 will not change, then the saving must change by the amount of the change in t_1, but in the opposite direction. More formally,

$$\Delta s = \Delta y_1 - \Delta t_1 - \Delta c_1$$
$$\Delta s = 0 - \Delta t_1 - 0$$
$$\Delta s = -\Delta t_1$$

So if the government reduces the taxes in the first period for each consumer by 1 unit, then the consumers will increase their saving exactly by 1 unit.

Discussion:
The Ricardian equivalence theorem is a useful starting point for thinking about the effects of government deficit. Governments can finance their deficit by taxing people or by borrowing (i.e. issuing debt). However, the government must eventually pay the debt by raising taxes in the future. The choice is therefore between "tax now" and "tax later". Our simple framework suggests that from the consumers' point of view there is no difference between "tax now" or "tax later". What matters for the optimal choice of consumption is the present value of the lifetime taxes, $t_1 + \frac{t_2}{1+r}$, but not the specific magnitudes of t_1 and t_2.

It is useful to list the assumptions of our two-period model, which are responsible for the Ricardian equivalence theorem.

1. We assume that all the households are taxed equally. In the real world it could be that the tax burden is not shared equally, so that tax policies will have an effect on the distribution of wealth in the economy.
2. In the model, the same people who receive a tax cut are the ones who have to pay the government debt in the future. In the real world the government can postpone the tax increase until long in the future, when consumers who received the tax cut are either retired or dead. In this case, the government tax policy will involve intergenerational transfer.
3. In the model the taxes were lump-sum. If we change this assumption and let the taxes be a fraction of income and also tax the interest rate earnings in the second period, then the timing of taxes will matter for the optimal choice of the household. To see this, notice that if the government taxes the interest earnings in the second period, then the household's net-of-taxes saving in the second period is $s(1+r(1-t_2))$. This means that the slope of the budget constraint is now $1+r(1-t_2)$, so if the government changes the timing of the taxes, the household's budget constraint will not remain unaffected.
4. Finally, in the model consumers can borrow or lend at the same interest rate as much as they please. If we relax these assumptions, the timing of taxes would affect the optimal consumption choices. A simple example can illustrate why. Suppose that some consumer is not allowed to borrow, so he is forced to consume his endowment. Changes in taxes will change the consumer's endowment, and therefore the consumption in both periods will change.

Appendix: Solving the consumer choice problem with Cobb-Douglas preferences
Suppose that the utility function has the form $U(c_1,c_2)=\ln c_1+\beta\ln c_2$, where $\beta>0$. This is a version of Cobb-Douglas preferences. The coefficient β is a weight on the utility from consumption in the second period. The greater β is, the more patient is the consumer. The consumer's problem that we need to solve is therefore

$$\begin{cases}\max\limits_{c_1,c_2}\ln c_1+\beta\ln c_2\\ s.t.\\ c_1+\dfrac{c_2}{1+r}=y_1-t_1+\dfrac{y_2-t_2}{1+r}\end{cases}$$

This is a standard problem, similar to the ones we have solved before.
The Lagrange function is

$$L=\ln c_1+\beta\ln c_2-\lambda\left[c_1+\frac{c_2}{1+r}-we\right]$$

First order conditions:

$$\frac{\partial L}{\partial c_1} = \frac{1}{c_1} - \lambda = 0$$

$$\frac{\partial L}{\partial c_2} = \frac{\beta}{c_2} - \frac{\lambda}{1+r} = 0$$

Thus the familiar condition of MRS = slope of the budget constraint is

$$\boxed{\frac{c_2}{\beta c_1} = 1+r}, \text{ or } c_2 = \beta(1+r)c_1$$

Plugging this into the budget constraint gives the demand for c_1

$$c_1 + \frac{\beta(1+r)}{1+r}c_1 = we$$

$$c_1(1+\beta) = we$$

$$c_1 = \left(\frac{1}{1+\beta}\right)we$$

The demand for c_2 is therefore

$$c_2 = \left(\frac{\beta}{1+\beta}\right)(1+r)we$$

Thus, the consumer spends a fraction $1/(1+\beta)$ of his lifetime income on first period's consumption and a fraction $\beta/(1+\beta)$ of his lifetime income on the second period's consumption. It makes intuitive sense that the higher the relative weight on a particular good in the utility, the greater is the demand for that good.

Substituting the expression of *we* in the demand gives the demand for c_1 and c_2:

$$\boxed{\begin{aligned} c_1 &= \left(\frac{1}{1+\beta}\right)\left(y_1 - t_1 + \frac{y_2 - t_2}{1+r}\right) \\ c_2 &= \left(\frac{\beta}{1+\beta}\right)\left((y_1 - t_1)(1+r) + y_2 - t_2\right) \end{aligned}}$$

The saving is then

$$s = y_1 - t_1 - c_1$$

$$s = y_1 - t_1 - \left(\frac{1}{1+\beta}\right)\left(y_1 - t_1 + \frac{y_2 - t_2}{1+r}\right)$$

$$s = (y_1 - t_1)\left(1 - \frac{1}{1+\beta}\right) - \left(\frac{1}{1+\beta}\right)\left(\frac{y_2 - t_2}{1+r}\right)$$

$$\boxed{s = (y_1 - t_1)\left(\frac{\beta}{1+\beta}\right) - \left(\frac{1}{1+\beta}\right)\left(\frac{y_2 - t_2}{1+r}\right)}$$

Notice that higher interest rate implies lower consumption in the first period, and higher saving and consumption in the second period. This makes intuitive sense, because $1+r$ is the relative price of current consumption in terms of future consumption. If interest rate goes up, the consumer will substitute future consumption for current consumption and therefore will save more in the first period.

Also, the higher the current net-of-tax income is, the greater is the saving, and the higher is the second period net-of-tax income, the lower is the saving. This makes intuitive sense. If the current income is relatively high, we would like to save more for the future, and if the future income is relatively high, we would like to save less for the future.

Finally notice that an increase in y_1 by 1 unit will increase c_1 by less than 1 unit (by $1/(1+\beta)$ to be precise). The saving will increase by $\beta/(1+\beta)$. Thus we see the consumption smoothing result here.