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Investment

1 Two-Period Model of Investment

1. There is one firm that can produce output in two periods according to

$$\begin{aligned} Y_1 &= A_1 K_1^{\theta} L_1^{1-\theta} \\ Y_2 &= A_2 K_2^{\theta} L_2^{1-\theta} \end{aligned}$$

where A_1, A_2 are productivity parameters (TFP - Total Factor Productivity), K_1, K_2 are the levels of physical capital in the two periods, and L_1, L_2 are labor inputs (number of workers employed by the firm).

2. The firm owns the capital stock in each period, and consumers own the firm. That is, the firm belongs to the shareholders, who are entitled to the stream of profits from the firm.

3. The capital stock evolves according to

$$K_2 = (1 - \delta) K_1 + I \tag{1}$$

where δ is depreciation rate of capital and I is investment in capital in the first period.

4. The capital in the first period, K_1, is exogenously given in the model, while the second period capital is a result of the firm's investment decision.

5. The firm decides on the labor demand in each period (L_1, L_2) and on the magnitude of the investment in the first period, I, and thus implicitly chooses K_2.

6. The distributed profit (dividends) in each period is given by

$$\begin{aligned} \pi_1 &= Y_1 - w_1 L_1 - I \\ \pi_2 &= Y_2 + (1 - \delta) K_2 - w_2 L_2 \end{aligned}$$

Notice that since the economy lasts for two periods, the firm can sell the nondepreciated capital stock, so the revenue in the second period is $Y_2 + (1 - \delta) K_2$.

1.1 Optimal investment decision

Recall that the firm is owned by the consumer and the lifetime value of the firm to the consumer is given by

$$V = \pi_1 + \frac{\pi_2}{1 + r}$$

where r is the real interest rate. Thus, the firm's problem is to choose L_1, L_2, I, K_2 to maximize the present value of the stream of dividends

$$\begin{aligned}
\max_{L_1,L_2,I,K_2} V &= A_1K_1^\theta L_1^{1-\theta} - w_1L_1 - I + \frac{A_2K_2^\theta L_2^{1-\theta} + (1-\delta)K_2 - w_2L_2}{1+r} \\
&\quad s.t. \\
K_2 &= (1-\delta)K_1 + I
\end{aligned}$$

Notice that choosing particular value of I essentially pins down K_2, we can let the firm directly choose K_2. Substitute the constraint into the objective and obtain the following firm's problem:

$$\max_{L_1,L_2,K_2} V = A_1K_1^\theta L_1^{1-\theta} - w_1L_1 - K_2 + (1-\delta)K_1 + \frac{A_2K_2^\theta L_2^{1-\theta} + (1-\delta)K_2 - w_2L_2}{1+r}$$

The first order conditions with respect to L_1 and L_2 are

$$\begin{aligned}
\frac{\partial V}{\partial L_1} &= 0, \Rightarrow (1-\theta)A_1K_1^\theta L_1^{-\theta} = w_1 \\
\frac{\partial V}{\partial L_2} &= 0, \Rightarrow (1-\theta)A_2K_2^\theta L_2^{-\theta} = w_2
\end{aligned}$$

This means that in each period the firm wants to hire labor up to the point where the marginal product of labor equals the wage. The first order condition with respect to K_2 is

$$\frac{\partial V}{\partial K_2} = -1 + \frac{\theta A_2K_2^{\theta-1}L_2^{1-\theta} + 1 - \delta}{1+r} = 0$$

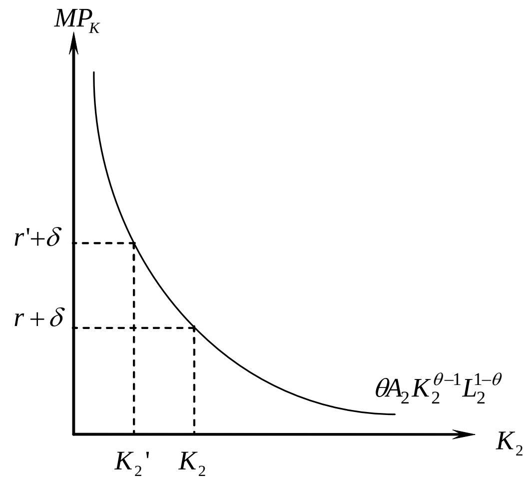
The interpretation is as follows. Increasing next period's capital by 1 unit costs 1 unit of current dividends. The benefit in the next period comes from two sources: (1) the revenue in the next period will increase by the marginal product of capital, $\theta A_2K_2^{\theta-1}L_2^{1-\theta}$, and (2) the nondepreciated capital can be sold in the next period and thus we have $1-\delta$ units of aditional revenue. The benefit from investment is collected in the second period, so it is discounted by $1+r$ to obtain its present value. Thus, the optimal level of K_2 (and therefore of I) is determined by the condition

$$\begin{aligned}
-1 + \frac{\theta A_2K_2^{\theta-1}L_2^{1-\theta} + 1 - \delta}{1+r} &= 0 \\
\theta A_2K_2^{\theta-1}L_2^{1-\theta} - \delta &= r \qquad (2)
\end{aligned}$$

which means that the real interest rate is equal to the marginal product of capital net of depreciation. This condition makes intuitive sense. In equilibrium the return in the financial market must be equal to the return in the capital market. If we invests 1 unit in the physical capital, the net return will be the marginal product of capital net of depreciation. If one invests in the financial market, the net return is r. In equilibrium there should be no arbitrage opportunities, and thus the rates of return must be equalized. It should make intuitive sense that an **increase** in r will **decrease** the investment in physical capital since the return in the financial market becomes relatively higher.

1.2 Changes in interest rate

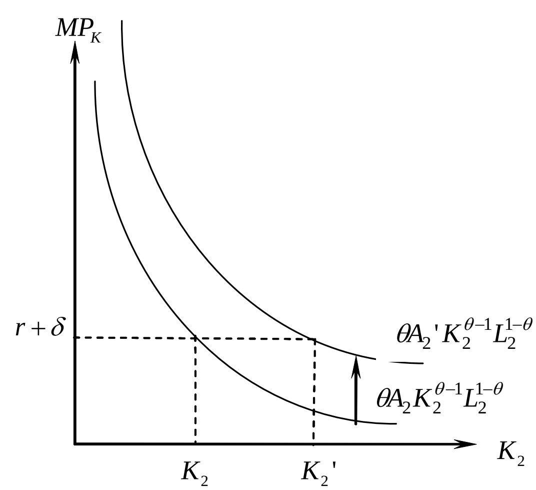
The next figure illustrates the effect of an increase in r on the optimal investment decision.



The downward slopping curve is the marginal product of capital MP_K. The optimal K_2 is obtained at the point where the marginal product of capital is equal to $r+\delta$. Higher real interest rate leads to lower K_2 and thus lower investment (since $I = K_2 - (1-\delta) K_1$). The intuition is as follows. The real interest rate represents the opportunity cost of investment in physical capital. Higher real interest rate means that the return in the financial market is higher, which makes the investment in physical capital less attractive. As shown in the figure, when interest rate goes up from r to r', the optimal future capital falls from K_2 to K_2'.

1.3 Changes in technology

The next figure illustrates the effect of an increase in A on the optimal investment decision.



Notice that the marginal product curve shifts upward, so that for any given level of K_2 its marginal product increases. The optimal level of K_2 (and also of investment) will therefore increase. The net return on investment in physical capital is the marginal product of capital (net of depreciation), hence it is intuitive that when the marginal product of capital goes up, the investment in physical capital should go up.

1.4 Solving for optimal investment

Solving equation (2) gives the optimal future capital K_2 as a function of real interest rate.

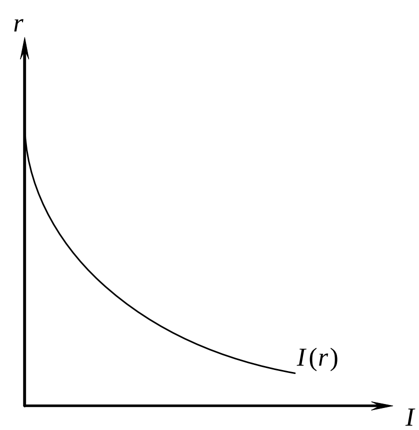
$$\begin{aligned} \theta A_2 K_2^{\theta-1} L_2^{1-\theta} &= r+\delta \\ \frac{\theta A_2 L_2^{1-\theta}}{r+\delta} &= K_2^{1-\theta} \\ K_2 &= \left(\frac{\theta A_2 L_2^{1-\theta}}{r+\delta}\right)^{\frac{1}{1-\theta}} \end{aligned}$$

Notice that from the above equation we clearly see that optimal K_2 is decreasing in interest rate r and increasing in the productivity level A. Now substitute this into equation (1) to find the optimal investment

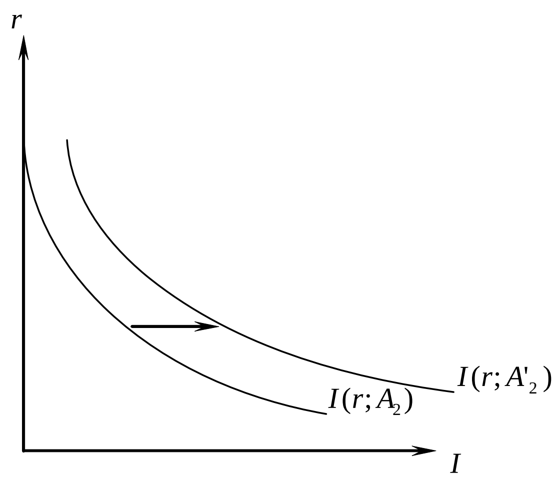
$$I = \left(\frac{\theta A_2 L_2^{1-\theta}}{r+\delta}\right)^{\frac{1}{1-\theta}} - (1-\delta) K_1 \quad (3)$$

Observe that the optimal investment is decreasing in r and decreasing in current capital (K_1). The intuition for the last observation is simple: if today's capital stock is big, we don't need to invest as much in order to attain the optimal level of future capital.

The next figure show the investment demand curve as a function of real interest rate.



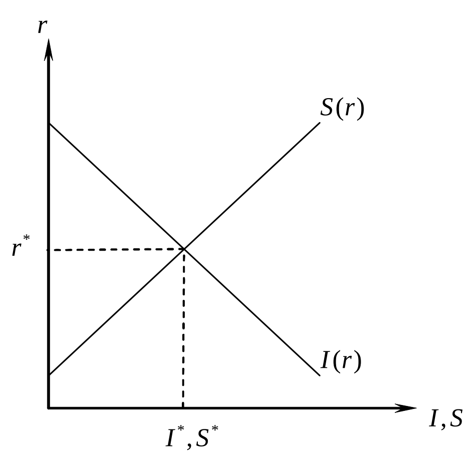
The price of investing in physical capital is the real interest rate - the opportunity cost of investment. Changes in the real interest rate is reflected in a movement along the same demand curve, but the curve itself *will not shift*. Changes in parameters other than r will shift the entire curve. For example, higher expected productivity ($A_2 \uparrow$) will shift the entire demand curve to the right. With higher future productivity the firm would like to invest more at any given real interest rate. The next figure shows the impact of ($A_2 \uparrow$) on the demand for investment.



In the above figure $A_2' > A_2$ results in shift to the right of the demand for investment. It is clearly seen from equation (3) that an increase in future productivity should increase investment.

2 Capital Market

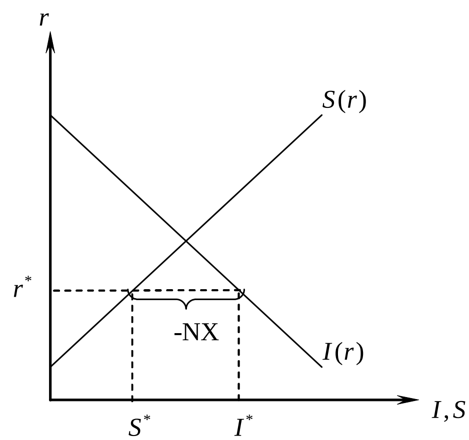
Now we are ready to put together the supply of saving with the demand for investment. We start with closed economy first. The next figure shows the capital market. For simplicity, we draw linear supply and demand curves.



The supply of saving curve is assumed upward slopping. Recall from our analysis of the saving decisions of household, we concluded that a *lender* will not necessarily increase his saving when the real interest rate goes up. Nevertheless, we assume that the substitution effect is stronger than the income effect, which ensures that the total saving of households is increasing in interest rate.

In a closed economy, national saving and domestic investment are equal. The above graph shows that the real interest rate and the amount of saving and investment is determined in equilibrium, at the intersection of supply and demand.

In an open economy, national saving can differ from domestic investment. For example, if the world interest rate is below r^* then the national saving will not be enough to finance the domestic investment, and the difference has to be borrowed. As we discussed before, borrowing from the rest of the world is the negative of trade deficit. The next figure illustrates an economy with $S < I$.

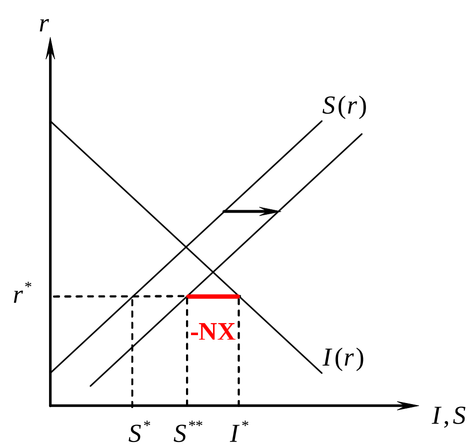


In this economy (as in the U.S. currently) the national saving falls short of the domestic investment, and the economy borrows the amount of $-NX$ (the trade deficit).

What affects the trade deficit according to our theory? It is obvious from the graph that national saving and domestic investment together determine the size of the trade deficit. In the next sections we will work with this model, to analyze the impact of different events on the saving, investment and the trade deficit in the economy.

2.1 Decline in government deficit ($S_G \uparrow$)

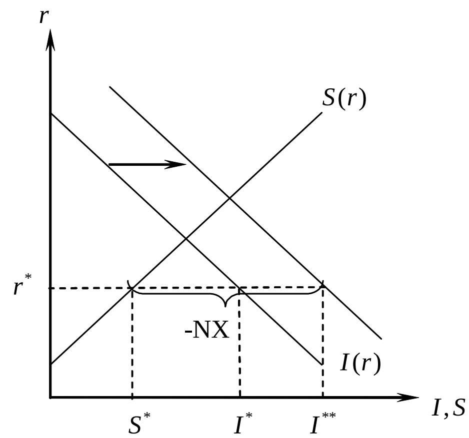
The total saving curve will shift to the right, as shown in the next figure.



For simplicity we assume that this change will not affect the world interest rate. We see that an increase in government saving leads to an increase the national saving and lowers the trade deficit. As bigger fraction of the domestic investment is funded by national saving, there is less borrowing from the rest of the world.

2.2 Increase future productivity at home ($A_2 \uparrow$)

The demand for investment curve will shift to the right, as demonstrated in the next figure.



If initially the national saving were not sufficient to fund the domestic investment, then after an increase in domestic investment the shortage is greater. The borrowing from the rest of the world is increasing, meaning higher trade deficit. The domestic investment increases, where all of the increase is funded by foreigners.

3 Summary

1. We showed that saving and investment in any economy are related. This relationship is called the **"Saving and Investment Equation"**.

2. We presented a **theory of saving** in a two period model of intertemporal choice. The model allows us to study the impact real interest rate, current and future income, and government tax policies on consumption and saving behavior. The model delivers an intuitive result, that people tend to smooth consumption. Another important result is Richardian Equivalence Theorem, which says that under certain conditions changes in government taxes do not affect the consumption and saving decisions of households.

3. We presented a **theory of Investment** decision by firms. The demand for investment is decreasing in real interest rate and increasing in future productivity.

4. Putting together the saving and investment theories, allows us to analyze the **capital market**. We demonstrated how real interest rate, the level of saving, investment and trade deficit are determined in the capital market.

4 Appendix: Firm With Unlimited Life Span

In these notes we analyzed the investment decision of firms that live for two periods. Now we show that the same condition for optimal investment holds when the firm lives unlimited number of periods. Suppose that the production function is more general than we used above, namely

$$Y_t = F(K_t, L_t; t)$$

The present value of the stream of profits is

$$V = \sum_{t=0}^{\infty} \frac{\pi_t}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{F(K_t, L_t; t) - w_t L_t - I_t}{(1+r)^t}$$

The firm's problem is

$$\begin{aligned} \max_{\{K_{t+1}, L_t\}_{t=0}^{\infty}} &= \sum_{t=0}^{\infty} \frac{F(K_t, L_t; t) - w_t L_t - I_t}{(1+r)^t} \\ & s.t. \\ K_{t+1} &= (1-\delta) K_t + I_t \end{aligned}$$

Substituting the constraint into the objective, gives

$$\max_{\{K_{t+1}, L_t\}_{t=0}^{\infty}} = \sum_{t=0}^{\infty} \frac{F(K_t, L_t; t) - w_t L_t - K_{t+1} + (1-\delta) K_t}{(1+r)^t}$$

F.O.C. with respect to K_{t+1} is

$$\begin{aligned} \frac{-1}{(1+r)^t} + \frac{F_1(K_{t+1}, L_{t+1}; t) + 1 - \delta}{(1+r)^{t+1}} &= 0 \\ F_1(K_{t+1}, L_{t+1}; t) + 1 - \delta &= 1 + r \\ F_1(K_{t+1}, L_{t+1}; t) &= r + \delta \end{aligned}$$

Thus, as before, the firm will invest up to the point where the marginal product of capital $F_1(K_{t+1}, L_{t+1}; t)$ is equal to the sum of the real interest rate and depreciation. Intuitively, the cost of investment is forgone interest rate and depreciation, and that has to be balanced by the return to investment - marginal product of capital.